Turbulence course, 2. week problem

1. Vector identities

When deriving the vorticity equation from the NSE we used two vector identities:

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} - \mathbf{B}(\nabla \cdot \mathbf{A}) + \mathbf{A}(\nabla \cdot \mathbf{B})$$

and

$$\frac{1}{2}\nabla \mathbf{u} \cdot \mathbf{u} = (\mathbf{u} \cdot \nabla)\mathbf{u} + \mathbf{u} \times (\nabla \times \mathbf{u})$$

prove them using the tensor notation: $\epsilon_{ijk}\partial_j \epsilon_{klm} A_l B_m = ...$ Remember: $\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

2. Solving the Poisson equation for pressure

The pressure is obtained from the NSE and incompressibility condition by applying the divergence operator on the NSE. The formal solution is:

$$p = \Delta^{-1}(\partial_i \partial_j u_i u_j).$$

For a periodic domain the velocity and the pressure can be expressed as Fourier-series:

$$\mathbf{u}(\mathbf{x}) = \sum_{\mathbf{k}} \mathbf{u}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}$$

and

$$p(\mathbf{x}) = \sum_{\mathbf{k}} p_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}.$$

Solve the Poisson equation for the pressure by finding an expression for $p_{\mathbf{k}}$.

3. The ABC flow

The ABC (Arnold-Beltrami-Childress) flow is defined as,

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} A\sin z + C\cos y \\ B\sin x + A\cos z \\ C\sin y + B\cos x \end{pmatrix} \tag{1}$$

Determine the pressure field in the ABC flow. Then show that the ABC flow is a static solution to the Euler equation.

The Euler equation is the Navier-Stokes equation without viscosity,

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p \tag{2}$$

The ABC flow is interesting because it is a regular static flow in which particle trajectories are chaotic. That is if a small particle at location $\mathbf{x}$ flows with fluid such that $d\mathbf{x}/dt = \mathbf{u}(\mathbf{x})$ it's motion is chaotic. (We do not show that here, but you can do that yourself on a computer).