Systems Driven By Alpha-Stable Noises Conference, La Jolla, California, May 17-20, 1998.

Ove Ditlevsen* and Peter Ditlevsen†

Introduction and abstract. It has almost become a standard in stochastic mechanics applications of stochastic differential equations that the driving forces are modeled as Gaussian white noises, that is, as scalar or vector Brownian motion increments. However, this modeling may not always lead to responses that comply well with observed data. In particular the tails of the observed response distributions may even for linear systems be more fat than the tails obtained for Gaussian white noise input. Also the excitation may show jumps that cannot be modeled by Gaussian white noise. The paper supports the possibility of using the larger class of so-called α-stable white noises (Lévy noises for $0 < \alpha < 2$, Gaussian white noise for $\alpha = 2$) to provide a better fit [4]. Lévy noise driven linear systems have responses that possess α-stable distributions with the same value of α as defined by the Lévy noise input. For $\alpha < 2$ the absolute moments exist only up to the order $\alpha-$, that is, the second order moment is infinite for any $\alpha < 2$ and the mean does not exist if $\alpha \leq 1$.

Alpha-stable noise and discussion of relevance. It is an elementary fact that any linear combination of n independent copies $X_1, \ldots, X_n$ of a Gaussian random variable X is a Gaussian random variable. One may ask whether also some non-Gaussian distribution types satisfy the condition $\forall a_1, \ldots, a_n \in \mathbb{R}\, \exists c \in \mathbb{R} : a_1X_1 + \ldots + a_nX_n =^d cX$, where $=^d$ means equal in distribution. A distribution with this property is said to be strictly stable. The characteristic function $\psi(u) = E[e^{iuX}]$ obviously must satisfy the condition $\prod_{j=1}^n \psi(a_j u) = \psi(cu)$. It is directly seen that a solution is $\psi(u) = \exp(-\sigma^\alpha |u|^\alpha)$ with $|c|^\alpha = |a_1|^\alpha + \ldots + |a_n|^\alpha$, $\sigma \geq 0$, and $0 < \alpha \leq 2$. It can be shown that $\psi(u)$ is a characteristic function only for $0 < \alpha \leq 2$ [3, 6], and that no other solutions exist. The parameter α is called the stability index and the corresponding random variables are said to be α-stable (in short: $S\alpha S$ for symmetric α-stable). In particular, for $\alpha = 2$ we obtain the Gaussian random variables of zero mean. The parameter σ is called the scale parameter. It is seen that cX has the scale parameter $|c|\sigma = (|a_1|^\alpha + \ldots + |a_n|^\alpha)^{1/\alpha}\sigma$. The variance of X is finite only for $\alpha = 2$ and is then equal to $2\sigma^2$. It can be shown [3, 6] that the tail probability $P(X > x)$ for $\alpha < 2$ asymptotically decreases proportional to $x^{-\alpha}$. For $\alpha = 1$ the tail probability is $P(X > x) = \frac{1}{2} - \frac{1}{\pi}\arctan\frac{x}{\sigma}$ (Cauchy distribution). No explicit expressions for $S\alpha S$-distribution functions exist for α different from 1 or 2. From the tail behavior it follows that $E[|X|^p] < \infty$ if and only if $p < \alpha$ when $\alpha < 2$.

The Brownian motion process (Wiener process) plays a fundamental role in standard stochastic dynamics as the formal source of the stochasticity of the excitation of almost any kind of system under study. The flexibility is large. In fact, the drift term and the diffusion term of a first order stochastic differential equation can in principle be constructed such that any prespecified marginal distribution of its stationary response is obtained. This includes distributions with power function tails down to the power $-(1+\epsilon)$ for any positive ϵ. For $\epsilon > 1$ the drift term may be chosen to be linear [1], in which case it is necessary to have a non-constant diffusion coefficient to get a non-Gaussian distribution. Distributions with power function tails turn out to be applicable modeling tools for representing observed data series e.g. in geophysics. This is also so for powers for which the distribution has no variance or even no mean. It may rightfully be claimed that real physical data measured on the earth come from a population with finite bounds. Surely, if the expectation, say, does not exist in a suggested theoretical model for such data, the expectation will in any case exist in the corresponding lower and upper truncated model. The point is that a model without mean reflects that the sequence of averages corresponding to an increasing sample of independent observations will exhibit fluctuations that stabilize towards the expectation the more slowly the smaller the truncation probability. The central limit theorem is valid, of course, but the convergence towards the normal distribution may be so slow that its predictions do not comply well with the empirical evidence. Taking the Cauchy distribution as an example, the average of any sample of any size has the same Cauchy distribution as the single observations.

¿From this it is clear that in itself the task of modeling fat distribution tails for the stationary solution process $X(t)$ to a stochastic differential equation does not motivate the generalization of the Brownian motion increment dB to the α-stable increment dL_α. The essential reason for making the generalization is that a convincingly good fit to the observed increment $dX(t)$ may be obtained only if a value of α less than 2 is used in the model. Moreover a model with $\alpha < 2$ gives sample functions that may exhibit jumps. For a dynamic system governed by a second order stochastic differential equation such jumps may take place in the velocity sample function. Thus these jumps correspond to force impulses generated directly dy the α-stable noise model.

The central limit theorem is often used as an argument for the application of the Brownian motion as the stochasticity source in stochastic mechanics. It is argued that Nature provides addition of many independent small random noise contributions of the same order of size, and that this leads asymptotically to Gaussian noise. However, this is only valid if the small contributions come from distributions of finite variance. In fact, the central limit theorem can be generalized to embrace the α-stable distributions as possible asymptotic distributions (theory of domains of attraction [3]). This also implies that it physically makes sense to work with coupled systems of two or more stochastic differential equations driven by a vector of α-stable noises with different values of $\alpha \in]0, 2]$.

Focusing on applications, it is unfortunate that the theory of α-stable noise driven systems is more complicated than in the Gaussian case $\alpha = 2$. The powerful tools of covariance and linear regression are not available. Lost is also the property that the square of the Brownian motion increment $dB(t)$ can be replaced by the deterministic time increment dt leading to the standard Itô calculus and the associated Fokker-Planck equation. However, the Chapman-Kolmogorov equation for Markov processes is valid also for $\alpha < 2$, and this equation gives a generalized Fokker-Planck equation for the characteristic functions of the response distributions obtained when the Brownian motion increment $dB(t) = dL_2(t)$ in the stochastic differential equation $dX = m(X,t) + \sigma(X,t)\, dB(t)$ is replaced by the Lévy noise $dL_\alpha(t)$ for $\alpha < 2$. Per definition the random increment $dL_\alpha(t)$ has the $S\alpha S$-distribution with the stability index α and the time increment $dt^{1/\alpha}$ as scale parameter. This Fokker-Planck equation is [5]

$$\frac{\partial \hat{p}(u,t)}{\partial t} = \int_{-\infty}^{\infty} [iu\, \hat{m}(v-u,t) - \frac{1}{\alpha}\hat{\sigma^\alpha}(v-u,t)|u|^\alpha]\hat{p}(v,t)\, dv \tag{1}$$

where $\hat{p}(u,t) = \int_{-\infty}^{\infty} p(x,t)e^{iux}\, dx$ is the characteristic function corresponding to the transfer probability density $p(x,t)$, $\hat{m}(u,t) = \lim_{c\to 0} \frac{1}{2\pi}\int_{-\infty}^{\infty} m(x,t)e^{-c^2x^2-iux}$ dx, and $\hat{\sigma^\alpha}(u,t) = \lim_{c\to 0} \frac{1}{2\pi} \int_{-\infty}^{\infty} \sigma^\alpha(x,t)e^{-c^2x^2-iux}\, dx$. The exponential factor $e^{-c^2x^2}$ is needed to ensure the existence of the Fourier integrals. Inverse Fourier transformation of (1) leads to the equation $\partial p(x,t)/\partial t = -\partial[m(x,t)p(x,t)]/\partial x + \frac{1}{\alpha}\partial^\alpha[\sigma^\alpha(x,t)p(x,t)]/\partial x^\alpha$, which for $\alpha = 2$ becomes the usual Fokker-Planck equation. The symbolic fractal derivative in the last term is defined as $\partial^\alpha f(x,t)/\partial x^\alpha \equiv -\frac{1}{2\pi}\int_{-\infty}^{\infty} |u|^\alpha \hat{f}(u)e^{-iux}\, du$ where $\hat{f}(u) = \int_{-\infty}^{\infty} f(x)e^{iux}\, dx$ $[= \int_{-\infty}^{\infty} \hat{\sigma^\alpha}(v-u,t)\hat{p}(v,t)$ dv for $f(x) = \sigma^\alpha(x,t)p(x,t)]$. For the even integer values $\alpha = 2(n-1), n \in \mathbb{N}$, this fractal derivative equals $(-1)^{n-1}$ times the usual derivative of order $2(n-1)$.

Explicit stationary solutions to (1) can be obtained for $\alpha = 1$ with $\sigma(x,t)$ independent of t and proportional to $x^{2(n-1)}, n \in \mathbb{N}$, and $m(x,t)$ independent of t and proportional to an odd degree polynomial in x with negative coefficient to the largest power. Formally (1) has an infinity of time invariant characteristic function solutions that are mixtures of a finite number of Cauchy distributions. The number of these and their parameters are determined exactly by the roots in a polynomial defined by $m(x)$ and $\sigma(x)$. However, only one of the mixtures actually fits with simulations of sample functions generated on the basis of the differential equation $dX = m(X) + \sigma(X)\, dL_1$. As indicated above the generalized Fokker-Planck equation associated to the equation $dX = e^{-c^2X^2}[m(X) + \sigma(X)\, dL_\alpha]$ is formally used to obtain a limit Fokker-Planck equation associated to the actual stochastic differential equation as obtained for $c \to 0$. Thus this puzzling ambiguity problem seems to come from applying a dubious limit operation. The authors have not found any mention of this behavior in the literature, and they do not know whether the same shows up for other values of $\alpha < 2$.

Considering that computer simulations provide conceptionally easy analysis also of systems driven by α-stable noise, the mathematically difficult issues of the topic should not prevent the inclusion of these stochastic processes in the modeling tool box of the applied sciences. For realization in the computer, the driving noise

dL_α can be approximated by a sequence of independent and identically distributed impulses, equidistantly separated in time by Δt. For any $\alpha \in]0,2]$ each impulse can be generated from the $S\alpha S$-distribution with scale parameter $(\Delta t)^{1/\alpha}$ by the formula $dL_\alpha = [(\Delta t)^{1/\alpha} \sin \alpha U/(\cos U)^{1/\alpha}][(\cos(1-\alpha)U)/W]^{(1-\alpha)/\alpha}$, where U and W are independent random variables, U with uniform distribution on $]-\pi/2, \pi/2[$, and W with exponential distribution of unit mean [4].

Geophysical example. An extensive vectorial data time series is available from a pointwise chemical analysis along an ice core drilled out from the top of the Greenland ice cap down to the base rock [2]. This ice core embraces about 250 kyr of snow fall. ¿From about 90 kyr to 10 kyr bp the data series covers the last glacial period with stationary appearance. The Ca variation measured with a resolution of one year may be taken as a so-called proxy for the climate variation. The series shows jumps between two populations corresponding to cold glacial periods and warmer interstadials, and statistical analysis shows that these jumps occur as points in a Poisson process with a mean waiting time of 1430 years between consecutive jumps [2]. To a good approximation the dynamics of the time series can be modeled by a pair of nonlinear first order stochastic differential equations driven by independent Brownian motion and Lévy noise with $\alpha = 1.75$ (together representing the short time scale weather phenomena such as strong storms). With $X = \log \mathrm{Ca}$, Y an auxiliary process, and a, b constants, these differential equations are $dX = -[dU(x)/dx]\, dt + a\, X\, dL_\alpha + b\, dY$, $dY = -Y\, dt + \sqrt{1+Y^2}\, dB$. The diffusion terms are obtained by comparing simulated increment distributions to the empirical increment distribution of the $\log \mathrm{Ca}$ data series. Attempts to use Brownian motion alone were not succesful. $U(x)$ is a numerically given potential function with two local minima. A first estimate of $U(x)$ was obtained from the stationary bimodal one-dimensional density function of the $\log \mathrm{Ca}$ data by use of the Fokker-Planck equation (1) [2]. This example shows that natural phenomena may very well be modeled by systems driven by Lévy noises.

References

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