A shell model of cascades in turbulent flow

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Understanding the turbulent nature of the atmospheric flow is still a subject of considerable scientific interest. The atmosphere is characterized by having energy on all scales of motion, from the 3 dimensional small scale boundary layer turbulence to the global scales of stationary quasi 2 dimensional planetary waves. One of the main difficulties in characterizing this flow is the lack of clearly separated spectral regimes, or spectral gaps in the spectrum, for the flow. The geostrophic and quasi-geostrophic flow of the atmosphere at the large scale was shown by Charney [2] to be equivalent to 2 dimensional flow. This classical description of the atmospheric flow works remarkably well due to the relatively stable stratification and the atmospheres small scale height. The characteristics of the 2 dimensionality of the flow is reflected in the energy spectrum. As an extension of Kolmogorovs, 1941(K41) [3] theory to the 2 dimensional case, Kraichnan [4] predicted from scaling arguments that the energy spectrum for 2 dimensional flow should scale with wave vector as $E_k \sim k^{-3}$. This was shown by Wiin-Nielsen [5], in an observational study, to be the case for the atmosphere. The result is remarkable, from the point of view of the energy transfer, in the sense that the main mechanism for generation of atmospheric waves, namely the baroclinic instability mechanism, is 3 dimensional in its very nature. Furthermore, a main forcing mechanism, release of latent heat in the tropics from cumulus convection, is small scale and also of a 3 dimensional nature. That is the main reason, why numerical forecasting is such a hard task in the tropics. The fact that quasi-geostrophic theory is not valid in the tropics is for a completely different reason, namely that the Coriolis force vanishes at the equator. From a forecasting point of view, quasi-geostrophy is obsolete; diabatic processes and divergencies are important for good forecasting.

The K41 hypothesis is that for 3 dimensional flow there will be a typical (large) scale of forcing where energy is inserted into the system. Energy is removed as heat production, at small scales by viscous dissipation. In between there will be a range of scales, the inertial range, in which the inertial motion of the flow is universally characterized by the scale represented by the wave-vector, k, and the mean dissipation per unit mass, η. From simple scaling arguments the energy spectrum then depends on these two parameters as $E_k \sim \eta^{2/3} k^{-5/3}$. This spectrum is seen in wind tunnels and all other experiments and observations of 3 dimensional flows. The energy is said to be cascaded from large to small scales. In terms of eddies and vortex tubes it can be expressed as smaller eddies feeding on larger ones, large structures break up into small ones. This will require a production of vorticity during that process. It is achieved by vortex tube stretching and bending. Thus a meteorological phenomenon such as the generation of tornados is a truly 3 dimensional phenomenon.

In 2 dimensions the situation is fundamentally different. For 2 dimensional flow the integral of the square of the vorticity, or in quasi-geostrophic flow the square of potential vorticity, the enstrophy, is conserved as well as the energy. In spectral space the enstrophy is given as $Z = \int dk k^2 E_k$, where $E = \int dk E_k$ is the energy. At the small scales the enstrophy will dominate and be dissipated. Consequently, there will be a cascade of enstrophy, and not energy, from the forcing scale through the inertial range to the dissipation scale. The energy spectrum obtained from the enstrophy cascade is the one mentioned above, $E_k \sim \eta^{2/3} k^{-3}$, where η is now the mean dissipation of enstrophy per unit mass. The energy, on the other hand, is confined to the large scales, where it is removed by linear drag. If both the integrals for Z and E are to be conserved in the inertial flow, the energy must be cascaded "up-scale". This is called the inverse or backward cascade of energy in 2 dimensional turbulence. The energy spectrum in this range should then scale with wave number as $k^{-5/3}$ as in the 3 dimensional case. This spectral behavior, attributed to the inverse energy cascade, has not yet been observed convincingly in the atmosphere. The creation of large scale coherent structures, vortices, seen in the atmosphere, maybe like blockings, might be attributed to the inverse energy cascade and thus be truly 2 dimensional phenomena.

A key issue for understanding the energy transfers in the atmosphere is to understand the energy spectrum and under which conditions the flow can be considered 2 dimensional and under which it is 3 dimensional. However, in numerical studies it is very difficult to simulate the cascade mechanisms, since this requires the full spectral range to be treated dynamically. In present day computer simulations of the Navier-Stokes equation for 3 dimensional flow resolutions of perhaps 1000 cubed points corresponding to Reynold numbers of 100-200 is at the limit. For 2 dimensional simulations somewhat higher resolution is possible, but in this case two inertial ranges, for forward enstrophy cascade and for backward energy cascades are required. In present atmospheric forecast models we have about 1 - 1.5 decades of inertial range flow.

As an alternative way to get insight into the cascading mechanisms in turbulent flow reduced wave-number models, which are truncated versions of the Navier-Stokes equation can be studied. In the rest of this short paper the Gledzer[6], Okhitani and Yamada[7] model, called the GOY model will be presented and discussed. We will here only focus on the model in a form supposed to mimic 2 dimensional turbulence.

The GOY model is an analogy to the Navier-Stokes equation in spectral form. The spectral domain is represented as shells, each of which is defined by a wavenumber $k_n = 2^n k_0$. So the variables are evenly spaced in $\log(k)$. We have $2N$ degrees of freedom, where N is the number of shells, namely the generalized complex shell velocities, u_n for $n = 1, N$. The reduced phase space enables us to cover a large range of wavenumbers, corresponding to large Reynold numbers. The dynamical equation for the shell velocities is,

$$\begin{aligned}\dot{u}_n &= ik_n(u^*_{n+2}u^*_{n+1} - \frac{\epsilon}{2}u^*_{n+1}u^*_{n-1} + \frac{(\epsilon-1)}{4}u^*_{n-1}u^*_{n-2}) \\ &\quad -\nu k_n^2 u_n - \nu' k_n^{-2} u_n + f\delta_{n,n_0},\end{aligned} \tag{1}$$

where the first term represents the non-linear wave interaction or advection, the second term is the dissipation, the third term is a drag term, and the fourth term is the forcing, where n_0 is some small wavenumber. The free parameter ϵ then determines the interaction coefficients of the non-linear advection term. The GOY model contains no information about phases between waves, thus there cannot be assigned a flow field in real space to the spectral field. The key issue for the behavior of the model is the symmetries and conservation laws obeyed. Depending on the value of ϵ the model has 2 inviscid invariants, which are quadratic in the shell velocities. In this paper we will concentrate on parameter values such that the 2 inviscid invariants are both positive definite. The invariants are then,

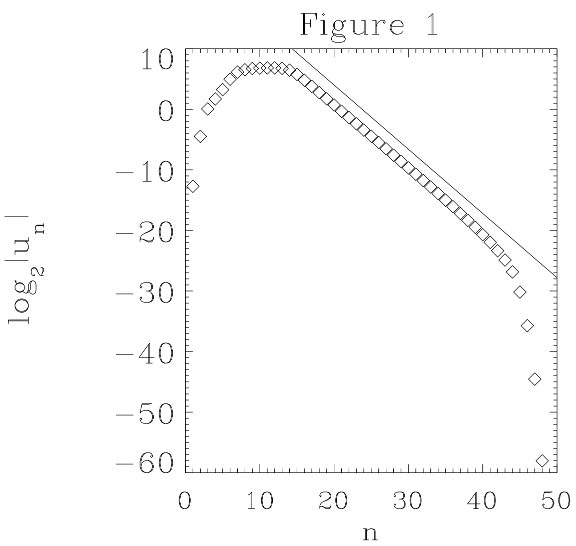
$$E = \frac{1}{2}\sum_{n=1}^{N}|u_n|^2, Z = \frac{1}{2}\sum_{n=1}^{N}k_n^\alpha |u_n|^2, \tag{2}$$

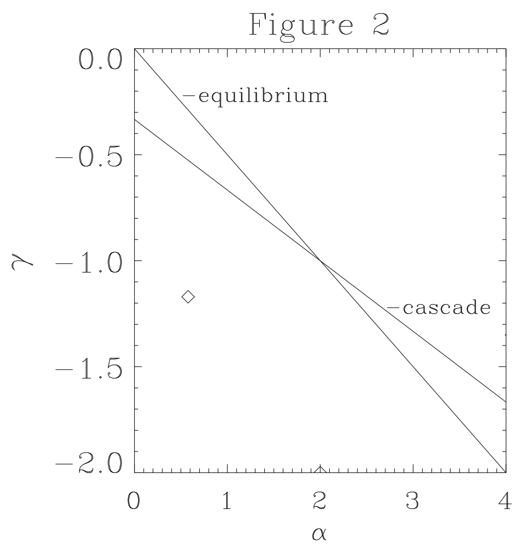
which should be thought of as the energy and enstrophy of the flow. The model in this form is then a model of 2 dimensional turbulence. The exponent α is related to the interaction coefficient, ϵ, as $\alpha = -\log_2(\epsilon - 1)$ [8]. So in the range $1 < \epsilon < 2$ we have $\infty > \alpha > 0$ and for $\epsilon = 5/4$ we have $\alpha = 2$ as is the case for the real turbulent flow.

The inertial sub-range is, as for real flow, defined as the range of shells where the forcing and dissipation are negligible in comparison with the non-linear interactions among shells. Since we apply the forcing at the small shell numbers and the dissipation at the large shell numbers, the inertial range (for forward cascade) is characterized by the cascade of enstrophy. The classical Kolmogorov scaling analysis can then be applied to the inertial range. Denoting η as the average dissipation of Z, this is then also the amount of Z cascaded through the inertial range. The spectrum of Z does, by the Kolmogorov hypothesis, only depend on k and η. From dimensional analysis we have, $[ku] = s^{-1}$, $[\eta] = [Z]s^{-1}$, $[Z] = [k^\alpha u^2] = [k]^{\alpha-2}s^{-2}$, and we get, $Z \sim \eta^{2/3}k^{(\alpha-2)/3}$. For the generalized velocity, u, we then get the "Kolmogorov-scaling",

$$|u| \sim \eta^{1/3}k^{-(\alpha+1)/3}. \tag{3}$$

This type of argument holds regardlessly of how well the model simulates real flow. Figure 1 shows the result of a numerical solution of (1). The absolute values of the shell velocities are averaged over 2×10^4 time units. They are plotted on a logarithmic scale as a function of shell number (0-50), which is the logarithm of k (base 2). The slope of the straight line is the scaling power, γ, such that $|u| \sim k^\gamma$. The system is forced at shell number 15, and the dissipation and drag dominates in the ranges $n \geq 40$ and $n \leq 5$ respectively.





A closed ergodic dynamical system will tend to a statistical equilibrium such that the energy and enstrophy will be distributed among the degrees of freedom with a temporal average for the energy given by $\overline{E_k} = (B + Ak^\alpha)^{-1}$. B^{-1} and A^{-1} are generalized energy - and enstrophy temperatures, corresponding to energy and enstrophy being conserved in time [9]. As a function of wave-number, k, there will then be two branches of the energy spectrum. For $k \ll 1$ we have $E_k \sim B^{-1}$ and equipartitioning of energy on the shells. For $k \gg 1$ we have $E_k \sim A^{-1}k^{-\alpha}$ corresponding to equipartitioning of enstrophy on the shells. In the latter case, which is the branch of forward cascade of enstrophy, the shell velocities will then scale as $|u| \sim k^{-\alpha/2}$. Returning to figure 1 we can now understand that the model has a second range of scaling, $5 < n < 15$, where we see an equipartitioning of energy.

The classical argument for having a cascade is that the system, when forced at the large scales, is out of equilibrium. It will then equilibrate by transferring enstrophy to small scales. The ultraviolet catastrophe is prevented by the viscous dissipation at small scales ($Re = \infty$ is a singular limit). The Kolmogorov scaling is characterized by an inertial range scaling with scaling power, $\gamma = -(\alpha + 1)/3$, and the statistical equilibrium is characterized by a scaling power, $\gamma = -\alpha/2$. For $\alpha = 2$ these coincide with $\gamma = -1$ [10], and we cannot distinguish the two. This coinciding scaling is a caviate of the GOY model not present in the real 2-D flow where the statistical equilibrium energy spectrum scales as k^{-1} and the cascade energy spectrum scales as k^{-3}. For all other values of α the scaling of the two cases are different. Motivated by this difference we ran the model with other values of α as well. Figure 2 shows the scaling power as a function of α. The diamonds represents the numerical results. All the slopes are steeper than or on the line representing the steepest slope of the two. The case $\alpha = 2$ is a borderline between the two descriptions. For $\alpha < 2$ the enstrophy is cascaded through the inertial range and for $\alpha > 2$ the enstrophy is equilibrated among the degrees of freedom of the inertial range. The enstrophy is transported through the inertial range by slow diffusion.

The two regimes corresponding to equipartitioning and cascade can be understood in terms of timescales for the dynamics of the shell velocities. A rough estimate of the timescales for a given shell n, is from (1) given as $T_n \sim (k_n u_n)^{-1} \sim k_n^{-1-\gamma}$. Again $\alpha = 2$, corresponding to $\gamma = -1$, becomes marginal where the timescale is independent of shell number. For $\alpha > 2$ the timescale grows with n and the fast timescales for small n can equilibrate enstrophy among the degrees of freedom of the system before the dissipation, at the "slow" shells, has time to be active. Therefore these models exhibit statistical equilibrium. For $\alpha < 2$ the situation is reversed and the models exhibit enstrophy cascades. Even though we have $\alpha = 2$ in real flows this analysis suggests that parameter choices $\alpha < 2$ might be more realistic than $\alpha = 2$ for mimicing enstrophy cascade in 2 dimensional turbulence.

Even though these types of models are very far from modeling the atmosphere, they display some features of real flow that are still not fully understood. We hope that from analyzing and understanding these features in a naive model, insight into the full problem can be gained.

References

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