

Cascades of energy and helicity in the GOY shell model of turbulence

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The effect of extreme hyperviscous damping, $\nu k_n^p, p=\infty$ is studied numerically in the GOY shell model of turbulence. It has recently been demonstrated [Leveque and She, Phys. Rev. Lett. **75**, 2690 (1995)] that the inertial range scaling in the GOY model is non-universal and dependent on the viscous damping. In the present study it is shown that the deviation from Kolmogorov scaling is due to the cascade of the second inviscid invariant. This invariant is non-positive definite and in this sense analogous to the helicity of 3D turbulent flow. © 1997 American Institute of Physics. [S1070-6631(97)02905-X]

The inviscid invariants, like the energy, of the Navier–Stokes equations are important quantities determining the dynamics of turbulent flow. The major difference between 2D and 3D flow originates from the existence of a second inviscid invariant, enstrophy, in 2D—absent in 3D flow. This gives strong constraints on the 2D flow dynamics. Conservation of enstrophy leads to the forward cascade of enstrophy and the backward cascade of energy in 2D, giving rise to large-scale coherent structures in the flow. In 3D there also exists a second quadratic inviscid invariant, namely the helicity, defined as the integral of the scalar product of the velocity and the vorticity. It has been proposed to be important for 3D turbulence.¹ It was shown by Kraichnan² that the interaction between waves of opposite helicity is stronger than interactions between waves of the same helicity. Consequently, helical flow might slow down the cascade of energy to the dissipation range.³

Since the spectral helicity density grows with a factor k , the modulus of the wave-vector, over the energy density the helicity dissipation would, in the $R \rightarrow \infty$ limit, prevent energy dissipation. The situation would resemble that of 2D turbulence where the cascade of enstrophy prevents the forward cascade of energy. In 2D turbulence the conservation of both $\int dk |u(k)|^2$ and $\int dk k^2 |u(k)|^2$ implies that the forward cascade of enstrophy must be accompanied by a backward cascade of energy. This strong constraint does, however, not apply in the case of helicity transfer since helicity is not positive definite. So we only have the weaker assessment mentioned above that the efficiency of energy dissipation could be depending on the non-linear helicity transfer and the helicity dissipation.

Considerable interest has lately been given to the GOY model of turbulence, introduced by Gletzer and examined by Yamada and Ohkitani.⁴ Comprehensive lists of recent references on the GOY model can be found in Refs. 5 and 6. The behavior of the helicity in different shell models have recently been investigated.^{6,7}

The GOY model is a simplified reduced wave-number analog to the spectral Navier–Stokes equations. The spectral domain is represented as shells, each of which is defined by a wavenumber $k_n = k_0 \lambda^n$, where λ is a scaling parameter defining the shell spacing. There are $2N$ degrees of freedom, where N is the number of shells, namely the generalized

complex shell velocities, u_n for $n=1, N$. The dynamical equation for the shell velocities is

$$\dot{u}_n = ik_n \left(au_{n+2}^* u_{n+1}^* + \frac{b}{\lambda} u_{n+1}^* u_{n-1}^* + \frac{c}{\lambda^2} u_{n-1}^* u_{n-2}^* \right) - \nu k_n^p u_n + f \delta_{n,n_0}, \quad (1)$$

where the first term represents the non-linear wave interaction or advection, the second term is the dissipation, and the third term the forcing, where n_0 is some small wavenumber. Throughout this paper the standard 3D GOY model parameter values, $\lambda=2$, $k_0=\lambda^{-4}$, $a=1$, $b=c=-1/2$ is used. The GOY model in this form contains no information about phases between waves, thus there cannot be assigned a flow field in real space.

The model has two conserved integrals, in the case of no forcing and no dissipation ($\nu=f=0$) (inviscid invariants). These are, $E = \frac{1}{2} \sum_{n=1}^N |u_n|^2$ and $H = \frac{1}{2} \sum_{n=1}^N (-1)^n k_n |u_n|^2$ which corresponds to the conservation of energy and a second non-positive definite quantity interpreted as analogous to helicity⁵ for the Navier–Stokes equation of 3D turbulence, hereafter referred to as the helicity. It should be stressed that this analogy is only in the sense that both quantities are non-positive definite. In this model each shell is maximally helical with alternating sign, since the numerical value of the helicity density is k times the energy density.

The forcing in (1) is applied at a small wavenumber and the dissipation dominates at large wavenumbers, so we can define an inertial range where the non-linear energy cascading terms dominate and the Kolmogorov scaling arguments apply. For an energy cascade we have the ‘‘Kolmogorov scaling’’ of the shell-velocities, $|u| \sim \eta^{1/3} k^{-1/3}$, and for a helicity cascade we have the ‘‘helicity scaling,’’ $|u| \sim \tilde{\eta}^{1/3} k^{-2/3}$, where η and $\tilde{\eta}$ are the mean dissipation per unit time of energy and helicity, respectively. The model has both the Kolmogorov scaling, $u_n = k_n^{-1/3} g(n)$ and the helicity scaling, $u_n = (-1)^n k_n^{-2/3} g(n)$ as unstable fixed points in the unforced and inviscid case. The function, $g(n) = g(n+3)$, is any mod(3) function. The mod(3) symmetry is an artifact of the GOY model discussed in detail in Ref. 5. It will become important in the following. The Kolmogorov fixed point plays an important role for the behavior of the GOY model, with forcing and dissipation, in the sense that the phase space

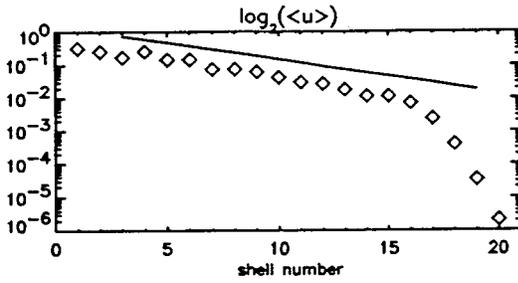


FIG. 1. The standard GOY model with $f_0=(1+i)\times 10^{-3}$, $n_0=4$ and $\nu=10^{-5}$. The model has 20 shells and is as before the run for 3000 time units after a 1000 time units spinup starting from the (unstable) Kolmogorov fixed point. $\log_\lambda(\langle |u_n| \rangle)$ is shown as a function of the shell number $n=\log_\lambda(k_n)$. The line indicates the Kolmogorov scaling with a spectral slope of $-1/3$.

trajectory of the shell velocities seems to “curl around” this point with the average values of the velocities close to the fixed point values. It was shown in a numerical study by Leveque and She⁸ that the inertial range scaling in the GOY model is not universal and is dependent on the form of the viscous damping. They attributed this non-universality to the reflection of energy flow from the viscous subrange back into the inertial range in the case of hyperviscosity ($p>2$). By studying the extreme case $p=\infty$, I suggest a different explanation for the non-universality of the inertial range scaling, namely that the cascade of helicity blocks the cascade of energy and thus changes the scaling. This effect is probably specific to the GOY model, and a consequence of the specific odd–even asymmetry of the helicity in the GOY model.

The numerical study is performed on two versions of the GOY model. First, the usual model with normal ($p=2$) dissipation and second, with viscosity only applied on the outermost shell, corresponding to $p=\infty$ hyperviscosity. The forcing in (1) is in both cases taken to be $f=f_0/u_{n_0}^*$ where f_0 is a constant. This gives a constant input of energy (and helicity) per unit time.

Figure 1 shows the result of a numerical calculation for $f_0=(1+i)\times 10^{-3}$, $n_0=4$ and $\nu=10^{-5}$. The model has 20 shells and is run for 3000 time units after a 1000 time units spinup starting from the (unstable) Kolmogorov fixed point. The first panel shows $\log_\lambda(\langle |u_n| \rangle)$ as a function of the shell number $n=\log_\lambda(k_n)$, where $\langle \cdot \rangle$ denotes the temporal average. The spectral slope is close to the Kolmogorov scaling, shown by the line, thus the model shows an energy cascade.

The sources and sinks for energy and helicity are

$$\dot{E} = \sum_n (u_n^* f_n - \nu k_n^2 u_n^2) \quad (2)$$

and

$$\dot{H} = \sum_n ((-1)^n k_n u_n^* f_n - (-1)^n \nu k_n^3 u_n^2). \quad (3)$$

This means that the helicity of opposite signs is dissipated at every second shell and therefore total helicity is produced for odd numbered shells in the dissipation range.

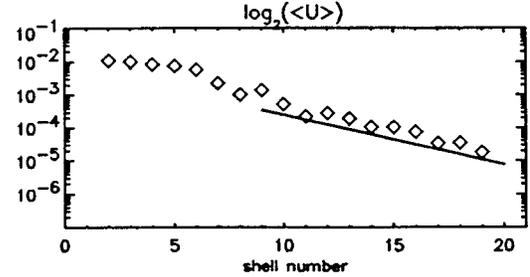
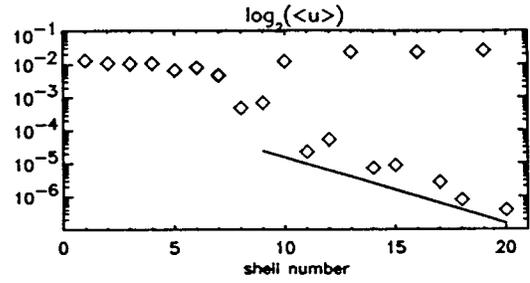


FIG. 2. The modified GOY model with $f_0=(1+i)\times 10^{-8}$ and $\nu_0=10^{-7}k_{20}^2=429.5$. This model has likewise 20 shells and is run for 3000 time units. The first panel is the same as in Fig. 1. The line in the figure has a slope of $-2/3$ corresponding to the cascade of helicity. Second panel shows the spectrum of $U_n=|u(n-1)u(n)u(n+1)|^{1/3}$. The line has a slope of $-1/2$ corresponding to the equipartition of helicity.

The inertial range flow is dominated by the non-linear transfer of the conserved quantities. The key assumption by Kolmogorov (1941) is that the inertial range flow does not depend on the specific form of the small scale viscous dissipation. This is not the case for the GOY model. In order to illustrate this point a study of a slightly modified GOY model is presented here. The dissipation is taken to be active only on the outermost shell, number $N=20$; this corresponds to hyperviscosity in the limit, $p\Rightarrow\infty$, with $\nu=\nu_0 k_N^{-p}$ and ν_0 held constant. With this choice of dissipation the behavior of the GOY model changes dramatically even though the non-linear terms in the governing equation remains unchanged.

Figure 2 shows the result of a numerical integration with the modified GOY model where $f_0=(1+i)\times 10^{-8}$ and $\nu_0=10^{-7}k_{20}^2=429.5$. The first panel shows the velocity spectrum where the mod(3) symmetry of the shell model becomes dominant. The scaling behavior of shells numbered 10,13,16,19 is different from the rest. The line in the figure has a slope of $-2/3$ corresponding to the cascade of helicity rather than energy.

However, if $U_n=|u(n-1)u(n)u(n+1)|^{1/3}$ is considered the mod(3) symmetry is eliminated and the inertial range scaling approximately reemerges as an arithmetic mean of the scaling behaviors for the shells $3n, 3n+1, 3n+2$ (Fig. 2, second panel). The line in the figure has a slope of $-1/2$ corresponding to equipartitioning of helicity. The hyper-viscosity of the model, only pulling out helicity through the (18, 19, 20) triad, cannot maintain a helicity cascade.

The result can be interpreted as a spectral bump at the end of the spectrum with an approximate equipartition of helicity. Similar results have been found by Borue and

Orszag⁹ for numerical simulations of the 3D Navier–Stokes equation. In this case the accumulation seems to be an effect of energy not being able to be transferred across the ultraviolet cutoff.

In the case ($f = \nu = 0$) the model has, discarding the boundary effects, besides the two scaling fixed points a periodic solution. It is easy to verify that the following satisfies the dynamical equation (1):

$$\begin{aligned} u_{3n-2}(t) &= s k_0^{-1} \lambda^{-3n}, \\ u_{3n-1}(t) &= \lambda^{3n\gamma} \sqrt{-\alpha_1} e^{-i\sqrt{-\alpha_1}\alpha_2 t}, \\ u_{3n}(t) &= \lambda^{3n\gamma} \sqrt{\alpha_2} e^{i\sqrt{-\alpha_1}\alpha_2 t}, \end{aligned} \quad (4)$$

with

$$\begin{aligned} \alpha_1 &= s \lambda^{-4} (1 + b \lambda^2 + c \lambda^{-3\gamma+1}), \\ \alpha_2 &= s \lambda^{3\gamma-3} (1 + b \lambda^{-3\gamma+1} + c \lambda^{-3\gamma+1}), \end{aligned} \quad (5)$$

and s is an arbitrary constant. The scaling parameter, γ , is related to the scaling fixed point by $\gamma = -(\alpha + 1)/3$, where $z = \lambda^\alpha$ is a solution to $1 + bz + cz^2 = 0$, thus a generator of one of the conserved quantities.¹⁰ So there are two values possible for γ ; $\gamma = -(2 + i\pi/\log(\lambda))/3$ corresponding to the generator, $z = -2$, of helicity, or the fluxless fixed point of the GOY model, and $\gamma = -1/3$ corresponding to the generator, $z = 1$, of energy, or the Kolmogorov fixed point.

There is in this periodic solution a complete phase locking of all the shells. The energy and helicity fluxes are both zero in the periodic solution, so in some sense it corresponds to the fluxless fixed point of the GOY model.

The shell-velocities of shells 8, 11, 14, 17, 20 are out of phase with those of shells 9, 12, 15, 18, while shell-velocities of shells 10, 13, 16, 19 are almost constant. From this it seems as if the periodic solution plays the same role for this modified GOY model as does the Kolmogorov fixed point for the GOY model. This could indicate the existence of an (unstable) limit cycle in the modified GOY model.

The dissipation of helicity is irregular, associated with the bursts in shell number 20 (and all other shells), the energy dissipation is completely blocked, resulting in a steady increase of energy concentrated on shells 10,13,16,19. The mechanism for preventing the forward cascade of energy in this model is analogous to the 2D case, where the forward cascade of enstrophy governs the dynamics.

The system shows no signs of approaching a statistical equilibrium simply because the helicity can grow to arbitrarily large negative values (dominated by shell number 19). The reason for this can be understood by examining the dissipative energy balance. In the case of normal viscosity it follows from (2) that the phase space trajectory will be attracted to the hyper-ellipsoid given by $\sum \nu k_n^2 |u_n|^2 = f$ which is a compact $2N - 1$ dimensional object. In the $p = \infty$ case this object will be the $2N - 1$ dimensional ‘‘hyper-cylinder’’ defined by $|u_{20}| = f/\nu k_{20}^2$. This is not compact and ergodic-

ity does not apply, thus the $p \rightarrow \infty$ is a singular limit and no statistical equilibrium can be reached. The relative change in energy over the integration is of the order 10^{-2} such that the system is in a quasi-equilibrium state where statistical equilibrium is reestablished in the limit where the energy injection rate goes to zero. Taking an even smaller forcing does not change the statistics, thus the statistical time averaging is meaningful.

If the energy cascade is effectively blocked the energy should be pulled out of the system by a drag at the small wavenumbers corresponding to the backward cascade in 2D. This is not seen, so an inverse energy cascade cannot be established. The model was also run with the usual dissipation but only active on every second shell. The result of this run was essentially the same as for the modified GOY model.

In conclusion we see that the non-linear transfer in the GOY model depends crucially on the dissipation properties for both conserved quantities, energy and helicity. It is still an open question how much the GOY model reflects the dynamics of the Navier–Stokes equation. These findings certainly indicate that the model is too restricted in some sense to represent real flow. The dissipative term introduced in the modified model is certainly not very realistic, in real flow helicity of both signs will be dissipated, which it is not in the modified model presented here. The findings from this simple model might indicate that in the inertial range flow the conservation of helicity could block the energy cascade and thus alter the Kolmogorov $k^{-5/3}$ scaling.

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- ¹J. C. Andre and M. Lesieur, ‘‘Influence of helicity on the evolution of isotropic turbulence at high Reynolds number,’’ *J. Fluid Mech.* **81**, 187 (1977).
- ²R. H. Kraichnan, ‘‘Helical turbulence and absolute equilibrium,’’ *J. Fluid Mech.* **59**, 745 (1973).
- ³W.-S. Wu, D. K. Lilly, and R. M. Kerr, ‘‘Helicity and thermal convection with shear,’’ *J. Atmos. Sci.* **49**, 1800 (1991).
- ⁴M. Yamada and K. Ohkitani, ‘‘Lyapunov spectrum of a chaotic model of three-dimensional turbulence,’’ *J. Phys. Soc. Jpn.* **56**, 4210 (1987); ‘‘The inertial subrange and non-positive Lyapunov exponents in fully-developed turbulence,’’ *Progr. Theor. Phys.* **79**, 1265 (1988); **60**, 983 (1988).
- ⁵L. Kadanoff, D. Lohse, J. Wang, and R. Benzi, ‘‘Scaling and dissipation in the GOY shell model,’’ *Phys. Fluids* **7**, 617 (1995).
- ⁶R. Benzi, L. Biferale, R. M. Kerr, and E. Trovatore, ‘‘Helical shell models for three dimensional turbulence,’’ *Phys. Rev. E* **53**, 3541 (1996).
- ⁷L. Biferale and R. Kerr, ‘‘On the role of inviscid invariants in shell models of turbulence,’’ *Phys. Rev. E* **52**, 6113 (1995).
- ⁸E. Leveque and Z.-S. She, ‘‘Viscous effects on inertial range scaling in a dynamical model of turbulence,’’ *Phys. Rev. Lett.* **75**, 2690 (1995).
- ⁹V. Borue and S. A. Orszag, ‘‘Self-similar decay of three-dimensional homogeneous turbulence with hyperviscosity,’’ *Phys. Rev. E* **51**, R856 (1995).
- ¹⁰P. D. Ditlevsen and I. A. Mogensen, ‘‘Cascade and statistical equilibrium in shell models of turbulence,’’ *Phys. Rev. E* **53**, 4785 (1996).