Dissipation in helical turbulence

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In helical turbulence a linear cascade of helicity accompanying the energy cascade has been suggested. Since energy and helicity have different dimensionality we suggest the existence of a characteristic inner scale, $\xi = k_H^{-1}$, for helicity dissipation in a regime of hydrodynamic fully developed turbulence and estimate it on dimensional grounds. This scale is always larger than the Kolmogorov scale, $\eta = k_E^{-1}$, and their ratio η/ξ vanishes in the high Reynolds number limit, so the flow will always be helicity free in the small scales. © *2001 American Institute of Physics.* [DOI: 10.1063/1.1404138]

In helical turbulence coexisting cascades of energy and helicity were envisaged by Brissaud *et al.*[1] Based on dimensional analysis it was conjectured that the helicity cascade is linear in the sense that the spectral helicity density follows the spectral energy density, $H(k) \propto E(k) \propto k^{-5/3}$. This scenario was supported numerically by André and Lesieur in an EDQNM closure calculation[2] and by Borue and Orzag[3] in a direct numerical simulation. Following Brissaud *et al.* the existence of a linear helicity cascade is due to an equal distortion time leading to the nonlinear transfer of energy and helicity. The distortion time at a scale k is estimated as[4]

$$\tau_k \sim \left(\int_0^k p^2 E(p) dp \right)^{-1/2}. \tag{1}$$

Here and in the following $\sim$ denotes "equal within order unity constants." [5] The nonlinear transfers of energy and helicity are then

$$\Pi_E(k) \sim kE(k)/\tau(k) \tag{2}$$

and

$$\Pi_H(k) \sim kH(k)/\tau(k). \tag{3}$$

From (1) and (2) the K41 result,

$$E(k) \sim \bar{\epsilon}^{2/3} k^{-5/3}, \tag{4}$$

follows where $\bar{\epsilon}$ is the mean energy dissipation or mean non-linear energy transfer or mean energy input. Correspondingly, from (1) and (3) we obtain

$$H(k) \sim \bar{\delta} \bar{\epsilon}^{-1/3} k^{-5/3}, \tag{5}$$

where $\bar{\delta}$ is the mean helicity input. The linear helicity cascade is derived under the assumption that helicity dissipation is negligible in the inertial range. The helicity density is $h = u_i \omega_i/2$, where $\omega_i = \epsilon_{ijk} \partial_j u_k$ is the vorticity. Conventionally the helicity is defined as $2h$, this is not important for the discussion presented here. An instructive way of representing this spectrally is to expand the velocity vector $u_i(\mathbf{k})$ in a basis of "helical modes." [6] The helical modes $\mathbf{h}_\pm$ are the

(complex) eigenvectors of the curl operator, $i\mathbf{k} \times \mathbf{h}_\pm = \pm k\mathbf{h}_\pm$. Using incompressibility, $\mathbf{k} \cdot \mathbf{u}(\mathbf{k}) = 0$, we have $\mathbf{u}(\mathbf{k}) = u_+(\mathbf{k})\mathbf{h}_+ + u_-(\mathbf{k})\mathbf{h}_-$ and the energy and helicity in the mode $\mathbf{u}(\mathbf{k})$ are

$$E(\mathbf{k}) = \mathbf{u}(\mathbf{k}) \cdot \mathbf{u}(\mathbf{k})^*/2 = (|u_+(\mathbf{k})|^2 + |u_-(\mathbf{k})|^2)/2 \tag{6}$$

and

$$H(\mathbf{k}) = \mathbf{u}(\mathbf{k}) \cdot \omega(\mathbf{k})^*/2 = k(|u_+(\mathbf{k})|^2 - |u_-(\mathbf{k})|^2)/2. \tag{7}$$

The spectral energy and helicity densities can then be separated into the densities of modes of positive and negative helicity $E(k) = E_+(k) + E_-(k)$ and $H(k) = H_+(k) + H_-(k) = k(E_+(k) - E_-(k))$. From this we have the rigorous constraint on the spectral helicity density,

$$|H(k)| \leq kE(k). \tag{8}$$

A similar constraint can be derived regarding the mean inputs of energy $\bar{\epsilon}$ and helicity $\bar{\delta}$. Suppose the flow is forced with a forcing $\mathbf{f}$ at the pumping scale such that $\mathbf{f}(\mathbf{k}) = 0$ for $|\mathbf{k}| > K$ where K is a wave number larger than the pumping scale. Then it follows that $|\bar{\delta}| \leq K\bar{\epsilon}$,[3] where K is a wave number at the pumping scale. When the scaling relations (4) and (5) are applied to the densities of positive and negative helicities separately, there must be a detailed cancellation of the leading scaling, such that

$$E_+(k) = (C/2)\bar{\epsilon}^{2/3} k^{-5/3} + (C_H/2)\bar{\delta}\bar{\epsilon}^{-1/3} k^{-8/3} \tag{9}$$

and

$$E_-(k) = (C/2)\bar{\epsilon}^{2/3} k^{-5/3} - (C_H/2)\bar{\delta}\bar{\epsilon}^{-1/3} k^{-8/3}, \tag{10}$$

where C and C_H are some (nonuniversal) order unity Kolmogorov constants.

The energy dissipation is given as $D_E = \nu \int_0^{k_E} k^2 E(k) dk$, and the upper limit of the integral which is the (inverse) Kolmogorov scale k_E is as usual determined by $\nu k_E^3 E(k_E) \sim \nu k_E^3 (\bar{\epsilon}^{2/3} k_E^{-5/3}) \sim \bar{\epsilon} \Rightarrow k_E \sim (\bar{\epsilon}/\nu^3)^{1/4}$. The dissipation is linear and can thus be split into dissipation of the positive and negative helicity parts of the spectrum separately. This implies that the dissipation of one sign of helicity

$(s=\pm)$ is $D_{H_s} \sim \nu \int_0^{k_H} k^2 H_s(k)dk = \nu k_H^4 E_s(k_H)$. The helicity of sign s is thus dissipated at a scale determined by

$$D_{H_s} \sim \nu k_H^4 E_s(k_H)$$

$$\sim \nu k_H^4(\bar{\epsilon}^{2/3}k_H^{-5/3} + s\,\bar{\delta}\bar{\epsilon}^{-1/3}k_H^{-8/3}) \sim \bar{\delta} \qquad (11)$$

and we arrive at an (inverse) inner scale k_H, different from the Kolmogorov scale k_E, for dissipation of helicity,

$$k_H \sim [\bar{\delta}^3/(\nu^3\bar{\epsilon}^2)]^{1/7}. \qquad (12)$$

It is easy to see that for any flow realization we must have $k_H \leq k_E$, so a pure helicity cascade is not possible. This result can also be obtained by estimating where the flow should be forced in order to dissipate the helicity at the Kolmogorov scale such that $k_H \sim k_E$. Pumping helicity into the flow at wave number κ implies $\bar{\delta} \sim \kappa\bar{\epsilon}$. We thus have

$$k_H \sim k_E \Rightarrow \left[\frac{(\kappa\bar{\epsilon})^3}{\nu^3\bar{\epsilon}^2}\right]^{1/7} \sim \left(\frac{\bar{\epsilon}}{\nu^3}\right)^{1/4} \Rightarrow \kappa \sim k_E. \qquad (13)$$

This shows that the the flow must be forced at the Kolmogorov scale which is in conflict with the existence of an inertial range. A similar result was obtained by Olla[7] in a different way using an argument based on the EDQNM approximation.

Furthermore, we have $k_H/k_E \propto \nu^{-3/7+3/4} = \nu^{9/28} \to 0$ for $\nu \to 0$. So again for high Reynolds number helical flow the small scales will always be nonhelical. The inner scale for helicity dissipation plays a different role in helical turbulence than the Kolmogorov scale. The dissipation of one sign of helicity at a given wave number will grow with wave number as $D_{H_s}(k) \propto k^{7/3}$, thus the dissipation of either sign of helicity will grow with wave number in the range $k_H < k < k_E$. This is only possible if there is a detailed balance be-

tween dissipation of positive and negative helicities in that range.

In conclusion, the scenario we propose for high Reynolds number helical turbulence is then the following. At the integral scale K energy and helicity is forced into the flow. In the inertial range $K < k < k_H$ there is a coexisting cascade of energy and helicity where helicity follows a "linear cascade" with a $H(k) \sim k^{-5/3}$ spectrum. In the range $k_H < k < k_E$ the dissipation of helicity dominates with a detailed balance between dissipation of positive and negative helicities and the right–left symmetry of the flow is restored. The balanced positive and negative helicities are generated in analogy to the enstrophy being generated in high Reynolds number flow. The proposed scenario has been illustrated in a shell model of turbulence.[8] However, since the considerations presented here are purely phenomenological they should be tested in experiments or numerical simulations.

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