A THEORETICAL PROOF OF THE EXISTENCE OF A CONSTANT VERTICAL RADIANCE ATTENUATION COEFFICIENT IN A HORIZONTALLY STRATIFIED OCEAN

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Abstract
A theoretical proof of the existence of an asymptotic radiance attenuation coefficient independent of \( \varphi \), \( \theta \), \( \theta_s \), and \( z \) is given below. \( \varphi \) is the azimuth angle, \( \theta \) the polar angle measured from zenith, \( \theta_s \) the solar angle in water defined by the Snellius law of refraction:

\[
\frac{\sin (90^\circ - h_s)}{\sin \theta_s} = \frac{n}{n_o},
\]

where \( h_s \) is the solar elevation and \( n \) and \( n_o \) are the refraction indices of sea water and air, respectively. The depth \( z \) is measured positively downwards.

Introduction
Different authors claim to have proved that the vertical radiance attenuation coefficient is reaching a constant value at asymptotic depth (see for instance FREISENDORFER, 1959, LUNDGREN and HØJERSLEV, 1971, and ZANEVELD and PAK, 1972), but all the proofs have in common, that some additional but plausible assumptions constraining the proofs have been made. Here an attempt is done to generalize and in the author's opinion to complete the proof.

Experimentally the existence of an asymptotic radiance attenuation coefficient was shown as early as in 1938 (JERLOV and LILJESTRÖM, 1938).

Theory
The various vertical attenuation coefficients in question are defined in the following way:
\( (1a) \quad K_E = - \frac{1}{E} \frac{\partial E}{\partial z} \) where \( E \) is the vector irradiance

\( (1b) \quad K_o = - \frac{1}{E_o} \frac{\partial E_o}{\partial z} \) where \( E_o \) is the scalar irradiance

\( (1c) \quad K_d = - \frac{1}{E_d} \frac{\partial E_d}{\partial z} \) where \( E_d \) is the downwelling irradiance

\( (1d) \quad K_u = - \frac{1}{E_u} \frac{\partial E_u}{\partial z} \) where \( E_u \) is the upwelling irradiance

\( (1e) \quad K_{od} = - \frac{1}{E_{od}} \frac{\partial E_{od}}{\partial z} \) where \( E_{od} \) is the downwelling scalar irradiance

\( (1f) \quad K_{ou} = - \frac{1}{E_{ou}} \frac{\partial E_{ou}}{\partial z} \) where \( E_{ou} \) is the upwelling scalar irradiance

\( (1g) \quad K_L = - \frac{1}{L} \frac{\partial L}{\partial z} \) where \( L \) is the radiance

\( (1h) \quad K_{\Phi} = - \frac{1}{L_{\Phi}} \frac{\partial L_{\Phi}}{\partial z} \) where \( L_\Phi \) is the path function

\( (1i) \quad L_\Phi(\theta, \varphi, z) = \int \frac{\beta(\theta, \varphi, \theta', \varphi', z) L(z, \theta', \varphi')}{4\pi} d\omega' \)

where \( \beta \) is the volume scattering function.

The notation used throughout this paper is the same as recommended in IAPSO Standard Terminology on the Optics of the Sea 1964.

It follows from the law of conservation of energy that the divergence of the irradiance vector in an absorbing medium is related to the absorption coefficient in the following way (GERSHUN, 1939)

\[ \text{div } E = \frac{1}{E_o} \frac{\partial E}{\partial z} \]

\[ (2) \]

It is now assumed that the horizontal divergence in (2) equals zero for all depths. In order to define this quantity at the surface a horizontal ocean level is assumed, which furthermore will be the only ocean boundary taken into consideration. The total attenuation coefficient \( c \), the absorption coefficient \( a \) and the scattering coefficient \( b \) are all assumed finite and positive. Finally only time independent radiance distributions at one particular wavelength are considered. The inherent properties \( a, b \) and \( c \) may vary in an upper stratum, but from a certain depth and below they shall remain constants.
These assumptions (or assumption of independency of p/c with depth) are necessary to give rise to an asymptotic radiance distribution (Preisendorfer, 1959).

From (1a), (2) and \( \text{div}_{\mathbf{r}} \mathbf{E} = 0 \) it is derived

\[
a = K_E \frac{E}{E_0}
\]  

(3)

\( 0 < b < \infty \) and \( 0 < a < \infty \) imply \( 0 < \frac{E}{E_0} < 1 \) as seen from the definitions of scalar and vector irradiance.

\( 0 < a < \infty \), (3), and the found limitation for \( \frac{E}{E_0} \) imply \( 0 < K_E < \infty \).

(1a), (1b), and (3) combine to give

\[
a(z) = K_E(z, \theta_s) \frac{E(0, \theta_s)}{E_0(0, \theta_s)} \cdot e^{\int_{0}^{z} (K_0(z, \theta_s) - K_E(z, \theta_s)) dz}
\]

(4)

The conditions \( a < \infty \) and \( \lim_{z \to \infty} a(z) = \text{constant} \) and (4) result in

\[
\lim_{z \to \infty} (K_0 - K_E) = 0 \quad \text{so that}
\]

\[
K_0(\infty, \theta_s) = K_E(\infty, \theta_s) \quad \text{for all} \ \theta_s
\]

(5)

Note that \( K_0 \to K_E \) for great depths at this stage does not imply that \( K_0(\infty, \theta_s) \) is independent of \( z \).

For great depths \( z \geq z_o \to \infty \) at which the absorption coefficient \( a \) is assumed constant equation (5) states

\[
- \frac{1}{E} \frac{dE}{dz} = - \frac{1}{E_0} \frac{dE_0}{dz}
\]

(6)

From integration of (6) one gets \( E = p(\theta_s) E_0 \), where \( p(\theta_s) \) is an arbitrary function of \( \theta_s \). Applied to (3) it is found

\[
a = \text{constant} = K_E(z, \theta_s) p(\theta_s)
\]

demonstrating that \( K_E \) is independent of depth for \( z \geq z_o \to \infty \) which combined with (5) implies that \( K_E = K_0 \) are functions of \( \theta_s \) only.

(3) can be rewritten

\[
a(z) = K_E \frac{\int_{0}^{4\pi} \frac{L \cos \theta \ d\omega}{4\pi \ L \ d\omega}} = K_E(z, \theta_s) <\cos \theta
\]

For \( z \geq z_o \) it follows
Now a consideration of an arbitrary radiance distribution is made at the depths \( z_1 \) and \( z_2 \), where \( z_2 > z_1 \), and \( z_1, z_2 > z_0 \).  

(7) signifies \( K_E(\theta_s) < \cos \theta = \text{constant} \), where the indices refer to the depths \( z_1 \) and \( z_2 \). 

Since \( K_E(\theta_s) = K_E(\theta_s) \), it follows 

\[
<\cos \theta_1 = <\cos \theta_2 > \quad (8)
\]

(8) suggests that the radiance distributions at \( z > z_0 \) are of similar shape for a fixed \( \theta \). An exact proof demonstrating that this is actually the case will be given below.

The classical equation of radiative transfer, having the previous assumptions in mind, can be written

\[
\frac{\partial L(z, \theta, \varphi, \theta_s)}{\partial r} = - c(z) L(z, \theta, \varphi, \theta_s) + L(z, \theta, \varphi, \theta_s) = \cos \theta \frac{\partial L(z, \theta, \varphi, \theta_s)}{\partial z} \quad (9)
\]

where \( r = z \sec \theta \).

(9) and (10) combine to give

\[
K_L \cos \theta = c - \frac{L(x)}{L} \quad (10)
\]

The path function can be written:

\[
L(z, \theta, \varphi, \theta_s) = \int_{4\pi} \beta(\theta, \theta', \varphi, \varphi') L(z, \theta', \varphi', \theta_s) \, d\omega' \quad (11)
\]

Denoting the absolute maximum and minimum scattering probability \( \beta_{\text{max}} \) and \( \beta_{\text{min}} \) respectively, one gets:

\[
0 < \beta_{\text{min}} E_0 < L(x, \theta, \varphi, \theta_s) < \beta_{\text{max}} E_0 < \infty \quad (11)
\]

For \( z > z_0 \to \infty \beta_{\text{min}} E_0 \) and \( \beta_{\text{max}} E_0 \) decrease in the same exponential manner which causes the path function \( L(z, \theta, \varphi, \theta_s) \) to do the same. From (10), (11), and the depth independency of \( K_o \) at great depths the following inequalities are established since \( L(z, \theta, \varphi, \theta_s) > 0 \):

\[
\left| c - \frac{L(z_0, \theta, \varphi, \theta_s) e^{-K_0(z-z_0)}}{L(z, \theta, \varphi, \theta_s)} \right| < c < \infty \quad (12)
\]

(12) demonstrates that \( L(z, \theta, \varphi, \theta_s) \) must decrease in the same exponential
manner as \( L(z, \vartheta, \varphi, \vartheta_s) \).

Consequently

\[
K_E \to K_0 \to K_L \to k(0) \quad \text{for} \quad z \to \infty \tag{13}
\]

where \( k(\vartheta_s) \) is a function of \( \vartheta_s \) only. Replacing \( K_L \) in (10) with \( k(\vartheta_s) \) the following expression is obtained for asymptotic depth:

\[
c - k(\vartheta_s) \cos \varphi = G(\vartheta_s) = \frac{L(z, \vartheta_s, \varphi, \vartheta_s)}{L(z, \vartheta_s, \varphi, \vartheta_s)} = \int \frac{L}{d\varphi} \tag{14}
\]

It can be shown that for all physical situations the last expression results in an asymptotic radiance distribution independent of \( \varphi \) (PREI-SENDORFER, 1959, LUNDQVIST and HØJERSLEV, 1971).

From the abovementioned and (13) one finds for \( z \geq z_0 \to \infty \):

\[
L(z, \vartheta, \vartheta_s) = L(z_0, \vartheta, \vartheta_s) e^{-k(z - z_0)} \tag{14}
\]

\[
L(z, \vartheta, \vartheta_s) = L(z_0, \vartheta, \vartheta_s) e^{-k(z - z_0)} (c - k \cos \varphi) \tag{15}
\]

(15) and (3) imply \( a < k < c \).

Two radiance distributions at depths \( > z_0 \) are now considered. In the following the indices 1 and 2 are referring to the radiance distributions \( L_1 \) and \( L_2 \) having \( \vartheta_s = \vartheta_s_1 \) and \( \vartheta_s = \vartheta_s_2 \), respectively. It is assumed that \( L_1 \) and \( L_2 \) are differently distributed resulting in, say

\[
\frac{E_2}{E_{o1}} < \frac{E_1}{E_{o1}} \quad \text{or} \quad K_{E_1} < K_{E_2} \tag{16}
\]

since the absorption coefficient \( a \) is constant. The inequality \( K_{E_1} < K_{E_2} \) and (13) signify that \( k_1 < k_2 \) which means that the radiance distribution \( L_2 \) is more prolate than \( L_1 \) since the \( c \)-value is unchanged. (PREIUR and MOREL, 1971). But this is in contradiction to the other inequality \( \frac{E_1}{E_{o1}} > \frac{E_2}{E_{o2}} \). Therefore it is concluded that \( L_1 = L_2 \) and accordingly \( k_1 = k_2 \) independent of \( \vartheta_s \).

By use of (14), (15), and the last obtained result the following
equations are established demonstrating that all the vertical attenuation coefficients are inherent properties:

\[ K_E = K_o = K_d = K_n = K_{od} = K_{on} = K_n = K_L = \text{constant} \]

It should be noted that the rate of convergence towards the asymptotic state may differ depending on \( \theta_s \) and the wavelength.

Application of (15) gives

\[ \frac{L(z, \theta, \theta_s)}{L(z, \theta, \theta_s)} = c - k \cos \theta \]

the last expression being a function of \( \theta \) only. Disregarding the depth dependence, which already has been examined, it is seen that the radiance has the following form at asymptotic depth:

\[ L(\theta, \theta_s) = L(\theta) F(\theta_s) \]

and similarly for the path function

\[ L_{x}(\theta, \theta_s) = L_x(\theta) F(\theta_s) \]

Remarks

a) In the asymptotic case it was found:

\[ L(\theta) = \frac{L_x(\theta)}{c - k \cos \theta} \] (16)

For isotropic scattering i.e. \( \beta \equiv \text{constant} = \frac{b}{4\pi} \)

\[ L_x = \int \beta L' \, d\omega' = \frac{b}{4\pi} E_0 \text{ independent of } \theta \]

As a result the radiance distribution is exactly an ellipsoid having excentricity equal to \( k/c \).

In a polydispersed medium like naturally occurring sea water, one finds a strong forward scattering. As a result the resultant asymptotic radiance distribution is altered slightly from an ellipsoid towards a "pear-shaped" figure. This may be explained from the definition of \( L_x(\theta) \). A pronounced forward scattering will produce a decreasing \( L_x(\theta) \) at least in a region between \( 0^\circ \) and \( 90^\circ \). Applied to (16)
we consequently have a decreasing path function multiplied by the elliptic function \((c - k \cos \theta)^{-1}\) which brings about the postulated "pear-shaped" radiance distribution.

b) A crude but illustrating example of the general trend of an \(E_d\) - curve versus depth will be given below:

The water is assumed homogeneous and \(0 \leq \theta_s < \theta_{cr}\) where \(\theta_{cr}\) is the critical angle in water defined by \(\sin \theta_{cr} = n/o\).

In the upper stratum of the ocean the light attenuation is dominated by absorption (JERLOV, 1968a, JERLOV and NYGÅRD, 1969). At greater depths one will have a combined effect of absorption and scattering resulting in a larger attenuation of the light field. In the asymptotic region an approach to the vertical of the radiance distribution takes place causing an increase of \(E/E_0\). As can be seen from the Gershun equation this involves a less \(K_E = K_d\) than at intermediate depths.

In the special case \(\theta_s = 0\) a decrease of \(K_E\) at great depths will not be observed.

Summary

It has been proved that at asymptotic depths the vertical radiance attenuation coefficient \(k\) is an inherent property. Furthermore it was shown that at these depths the radiance distribution \(L\) had the following form:

\[
L(z, \theta, \theta_s) = L(z_0, \theta) f(\theta_s) e^{-k(z - z_0)}
\]

the path function \(L_p\) having the same \(z\) and \(\theta_s\) dependence.

Knowledge of the shape of the radiance distribution \(L\) and its first derivative with respect to \(z\) at asymptotic depth implies that \(k = K_E\) can be found and integration of \(L\) and application of the Gershun equation determines the absorption coefficient \(a\). Conversely, given the absorptance \(a\) and the radiance distribution \(L\) the attenuation coefficient \(k\) can be determined.

At asymptotic depth it was found:

\[
L_p(\theta) = (c - k \cos \theta) L(\theta) = 2\pi \int_0^\pi \rho(\theta, \theta') L(\theta') \sin \theta' d\theta'
\]
or
\[
\int \beta(\alpha) L(\theta') d\omega' - L(\theta) \int \beta(\theta) d\omega = (a - k \cos \theta) L(\theta)
\]

The last equation being derived from
\[
a + b = c \quad \text{and} \quad b = \int \beta d\omega
\]

\(\alpha\) is given by:
\[
\cos \alpha = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos \varphi
\]

Since \(L\), \(a\) and \(k\) are known the volume scattering function \(\beta\) can be determined in principle, which can be seen from the following reasoning: From mathematical theory it can be shown that \(\beta(\theta)\) can exactly be represented as \(\sum_{n=0}^{\infty} c_n P_n(\cos \theta)\) where \(P_n(\cos \theta)\) is the Legendre polynomial of degree \(n\).

Likewise the radiance distribution \(L\) can be expanded in terms of Legendre polynomials. Applied to (17) this allows an evaluation of \(\beta\). Knowing the scattering function the scattering coefficient and the total attenuation coefficient are straightforward determined (see for instance ZANEVELD and PAK, 1972).

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