A Note on Hydrodynamic Turbulence

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October 12, 2000

A lecture note for Fysik 712: Topics in Physics of Complex Systems

This note presents a very short review of some of the main characteristics and unknowns of turbulence. Turbulence is the chaotic and apparently random flow of a stirred fluid. Fluid flows vary a lot depending on the boundaries containing the flow, the stirring and the heating. The flow in the atmosphere of a rotating planet is different from the convection in a pot of boiling water. However, as long as the length scales in the flow are small in comparison to the largest scales determined by the boundaries and large in comparison to the molecular scales all flows seem to have common characteristics. Turbulence is these common characteristics of the flows.

1 The Navier-Stokes equation

Fluid mechanics is the description of fluids on scales large in comparison to the mean free path length of the molecules constituting the fluid. In this limit the fluid is regarded a continuum characterized completely by a velocity field $u_i(\mathbf{x}, t)$, a temperature field $T(\mathbf{x})$ and a density $\rho(\mathbf{x})$. At each point x_i the fluid is then fully characterized by the six field variables, 3 components of velocity, pressure, temperature and density. In order to determine the evolution of these we need 6 equations. These are derived from momentum conservation, mass conservation, energy conservation and the equation of state. In any concrete setting some of the field variables might be approximately constant and the number of equations reduced. When considering fully developed turbulence the fluid is traditionally regarded incompressible, which is a rather good approximation for water and air. This immediately eliminates two equations specifying density and temperature from considerations. The dynamics of such a fluid is described by the Navier-Stokes equation (NSE)

$$\partial_t u_i + u_j \partial_j u_i = -\partial_i p + \nu \partial_{jj} u_i + f_i \tag{1}$$

and the continuity equation

$$\partial_i u_i = 0. \tag{2}$$

The NSE describes the momentum conservation. This just states that the acceleration of a fluid particle equals the sum of the forces acting on the fluid particle (per unit mass). The left hand side is the material derivative of the velocity field, where the second term is the advection. The first term on the right hand side is the pressure gradient force, the second is the viscous friction (viscosity) and the last term represents all other forces. The last term is, strictly speaking, not a part of the NSE but we will not make to much of a fuss about that. The continuity equation is the equation for conservation of mass, where in the case of an incompressible fluid the density does not appear. The inverse of the density, which normally appears in front of the gradient of the pressure in (1) is thus also absorbed in the units of the pressure. From these (four) equations together with boundary – and initial conditions the three components of the fluid velocity u_i and the pressure p can in principle be determined. However, no general solutions to the NSE are known and only for very simple laminar flows can a solution be found. The pressure can immediately be eliminated from the NSE using the continuity equation (2). Assuming the forcing to be rotational $\partial_i f_i = 0$, we obtain a Poisson equation for the pressure by applying the divergence operator to the NSE,

$$\partial_{ii} p = -\partial_i u_j \partial_j u_i \tag{3}$$

which we can formally solve by applying the inverse Laplacian,

$$p = -\partial_{kk}^{-1} (\partial_i u_j \partial_j u_i). \tag{4}$$

We will make sense of the inverse of a differential operator when expressing the NSE in terms of Fourier components. The NSE can be brought on dimensionless form by defining

$$x = L\tilde{x},\ u = U\tilde{u},\ t = (L/U)\tilde{t} \tag{5}$$

where L is the length scale of the largest variations in the flow. L would typically be the size of the container or basin for a bounded flow or the size of an obstacle in an extended uniform flow. U is the typical velocity at this length scale. We can think of U as the typical velocity when coarse graining the flow at the length scale L. From L and U we can build a time scale which is simply the time it takes the fluid at uniform velocity U to travel the distance L. Inserting this into (1) and dropping the tilde gives the NSE on dimensionless form,

$$\partial_t u_i + u_j \partial_j u_i = -\partial_i p + \mathsf{Re}^{-1} \partial_{jj} u_i + f_i \tag{6}$$

where we have defined the dimensionless Reynolds number,

$$\mathsf{Re} \equiv \frac{UL}{\nu}, \tag{7}$$

and absorbed a factor U^2/L into the forcing. The pressure gradient term is, as can be seen from (4) dimensionally the gradient of a velocity squared so it scales as the advection term. All terms except for the viscosity are now of order unity. The viscosity is of the order of the inverse of the Reynolds number, so that the Reynolds number is a measure of the relative importance of the viscosity in comparison to the non-linear terms (the advection and the pressure gradient term) at the length scale L and velocity scale U. The Reynolds number is the fundamental characteristic of any given flow. For a Reynolds number smaller than one the flow will be quickly damped by viscosity. The viscosity acts as a smoother of irregularities and has the form of a diffusion term. When the Reynolds number becomes larger the flow will be more and more dominated by the non-linear terms.

Figures (1) and (1) shows the flow past a cylinder with increasing flow velocity and thus increasing Reynolds numbers.

For small Reynolds numbers the flow is smooth and regular. As the Reynolds number is increased the fluid motion in the wake becomes more

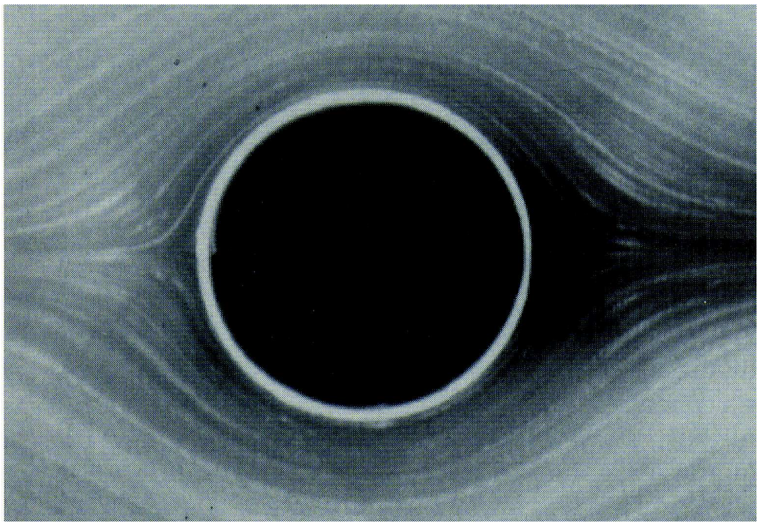

Figure 1: Flow past a cylinder at Reynolds number 0.16 [Vandyke,1982]

and more irregular. The successive symmetry breaking seen in the sequence is described in detail in Frisch' book [4]. In the last frame for very high Reynolds numbers the flow in the wake is completely chaotic and apparently random. This is what we call 'fully developed turbulence', and it characterizes many systems in Nature such as the flow in the atmospheric boundary layer, river flow, the wake after a jet-engine, smoke from a cigar and many other phenomena. All the richness of the complex behavior of these systems is, we believe, described by the NSE. However, no general theory exists from which we can relate directly the NSE and the rich phenomenology observed in Nature and experiments. The common phenomenology of turbulence is attributed to Richardson (1922) where he describes the flow at large scales to break up into smaller swirls, which again break up into yet smaller swirls until finally the swirls are so small that they are smoothed out, or dissipated, by the viscosity. The energy is inserted into the flow at large scales, it then cascades into smaller and smaller scales until it leaves the flow at the viscous scale. This lead in 1941 Kolmogorov to develop a phenomenological theory of turbulence.

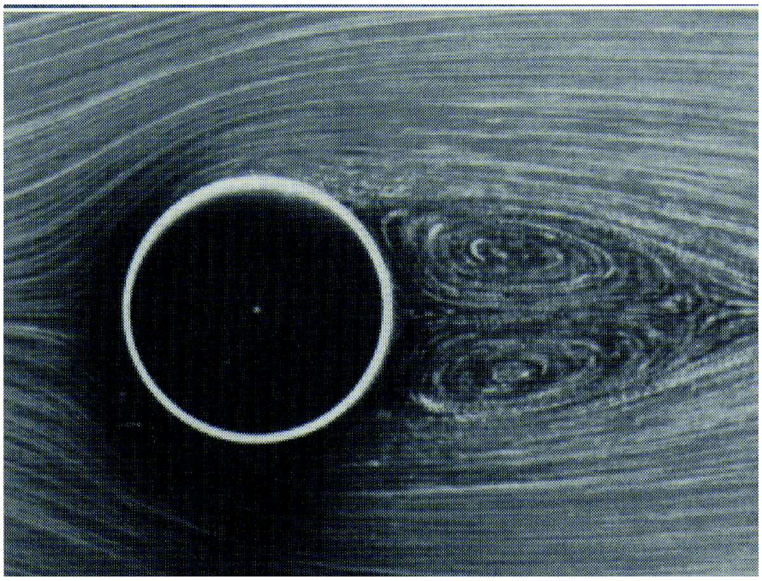

Figure 2: Flow past a cylinder at Reynolds number 26 [Vandyke,1982]

2 Kolmogorov's 1941 Theory (K41)

The paper [2] in which Kolmogorov presents the theory is only one half page long and the idea is very simple. It is presented thoroughly in Landau and Lifshitz' Fluid Mechanics book [5]. Here we will go through it briefly. Kolmogorov imagined a flow initiated by vigorous stirring and then left alone to slowly damp out by the viscosity. This is today denoted decaying turbulence. The flow is assumed to be homogeneous (translational invariant in the mean) and isotropic (rotationally invariant in the mean). The picture we will have in mind here is a flow maintained by a forcing active on large scales of the flow, such that the flow is in a state of statistical equilibrium in the sense that on average the energy input by the forcing is balanced by the energy dissipated by the viscosity (heating the fluid). The state of the flow is then characterized by the mean energy dissipation $\overline{\varepsilon}$ due to viscosity. The velocity characteristic of a given length scale $l << L$ is the typical velocity difference $\delta u(l) = |u(r+l) - u(r)|$, where we suppress all vector indices. This velocity difference is characteristic for the velocity associated with an eddy of size l. The effect of the larger scale flow velocity is merely to move or sweep the eddy through the flow as a rigid body. Likewise, if we consider a much smaller eddy within the larger eddy, the effect of the larger eddy on the smaller is the same as the effect of the larger scale flow on the large eddy. Since there is nothing physically special about a given length scale l in the flow we assume the flow to be self-similar in the sense that when $l_1 < l_2 << L$ the velocity differences are related by $\delta u(l_2) = f(l_1/l_2)\delta u(l_1)$, where f is some universal function. This implies that the velocity difference $\delta u(l)$ can only be a function of the scale l and the mean energy dissipation $\overline{\varepsilon}$. From dimensional counting, the only possible relationship is,

$$\delta u(l) \sim (\overline{\varepsilon} l)^{1/3}, \tag{8}$$

where $\sim$ means proportionality.

We shall use these kinds of arguments throughout this book. The scaling relation (8) is obtained from the fact that only quantities of the same dimension can be compared. So if we want to establish a functional relationship,

$$\delta u(l) = f(l, \overline{\varepsilon}) \tag{9}$$

the dimension on the right hand side must be the same as the dimension on the left hand side. Furthermore, the numerical value of the quantity on the left hand side cannot depend on change of units of the quantities on the right hand side which leave the units on the left hand side unchanged. If, for example, we measure length in millimeters and time in milliseconds in stead of meters and seconds, the numerical value of velocity is unchanged. However, the numerical values of l and $\overline{\varepsilon}$ changes. If (9) is to hold regardless of the change of units, f can only depend on the combination of l and $\overline{\varepsilon}$ which has the same dimension as the left hand side. The dimensions are $[\delta u] = m/s$, $[l] = m$, $[\overline{\varepsilon}] = m^2/s^3$, so for $[\delta u] = [l]^\alpha[\overline{\varepsilon}]^\beta$ we get, $\beta = \alpha = 1/3$. From this we get $\delta u(l) = \tilde{f}((\overline{\varepsilon} l)^{1/3})$. By changing the units on velocity, say by scaling length by a factor λ we get

$$\lambda \delta u(l) = \lambda f((\overline{\varepsilon} l)^{1/3}) = f(\lambda(\overline{\varepsilon} l)^{1/3}). \tag{10}$$

Thus we see that f must be a linear function and we obtain (8).

The relation (8) contains all the essentials of the K41 theory. The scale η at which the dissipation becomes important is called the Kolmogorov - or inner scale in contrast to L called the outer – or integral scale . From (1) we can get an estimate of the rate-of-change of the energy per volume due to dissipation at the scale η, $\overline{\varepsilon} \sim \nu u \Delta u \sim \nu \delta u(\eta)^2/\eta^2$. Using (8) we get,

$$\eta \sim (\overline{\varepsilon}/\nu^3)^{-1/4}. \tag{11}$$

So keeping the integral length scale velocity fixed, the Kolmogorov scale depends on the Reynolds number as $\eta \sim \mathsf{Re}^{-3/4}$.

The mean of the square of the velocity difference is called the second order structure function and is obtained by simply squaring (8),

$$S_2(l) \equiv \langle \delta u(l)^2 \rangle \sim (\overline{\varepsilon} l)^{2/3}. \tag{12}$$

The mean $\langle . \rangle$ denotes an ensemble mean, whereby we mean the average over many realizations of the flow with different initial conditions (drawn from some distribution). Assuming ergodicity this could as well be a temporal mean (in a given set of points), or a spatial mean in the case of homogeneity. We will freely assume these three to be equal or use either at our convenience without dwelling more into subtleties regarding the assumption of ergodicity or the distribution of initial conditions. A nice discussion can be found in Frisch' book.

3 The spectral Navier-Stokes equation

Consider the Fourier transform of the velocity field,

$$u_i(\mathbf{x}) = \int e^{i\mathbf{k}\mathbf{x}} \hat{u}_i(\mathbf{k}) d\mathbf{k}, \tag{13}$$

$$\hat{u}_i(\mathbf{k}) = \frac{1}{(2\pi)^3} \int e^{-i\mathbf{k}\mathbf{x}} u_i(\mathbf{x}) d\mathbf{x}. \tag{14}$$

We can express the NSE by inserting the Fourier transform of $u_i(\mathbf{x})$ in (1), moving the advection term to the right hand side,

$$\begin{aligned}\int e^{i\mathbf{k}'\mathbf{x}} \partial_t \hat{u}_i(\mathbf{k}') d\mathbf{k}' = &- \int \int e^{i\mathbf{k}'\mathbf{x}} \hat{u}_j(\mathbf{k}') \partial_j e^{i\mathbf{k}''\mathbf{x}} \hat{u}_i(\mathbf{k}'') d\mathbf{k}' d\mathbf{k}'' \\ &- \int \partial_i e^{i\mathbf{k}'\mathbf{x}} \hat{p}(\mathbf{k}') d\mathbf{k}' \\ &+ \int [\nu \hat{u}_i(\mathbf{k}') \partial_{jj} + \hat{f}_i(\mathbf{k}')] e^{i\mathbf{k}'\mathbf{x}} d\mathbf{k}'. \end{aligned} \tag{15}$$

By applying the operator $\int e^{-i\mathbf{kx}} \cdot d\mathbf{x}/(2\pi)^3$, using the orthogonality of the Fourier modes and performing the differentiations with respect to $\mathbf{x}$ we get,

$$\partial_t \hat{u}_i(\mathbf{k}) = \frac{1}{(2\pi)^3} \int\int \delta(\mathbf{k}' + \mathbf{k}'' - \mathbf{k})(ik_j'')\hat{u}_j(\mathbf{k}')\hat{u}_i(\mathbf{k}'')d\mathbf{k}'d\mathbf{k}'' - ik_i\hat{p}(\mathbf{k}) \\ -\nu k^2 \hat{u}_i(\mathbf{k}) + \hat{f}(\mathbf{k}). \quad (16)$$

The Fourier transform of the pressure is easily obtained by solving the Poisson equation (3) for an incompressible fluid, $\partial_{ii}p = -\partial_i u_j \partial_j u_i$. This can be done in spectral form exactly in the same way as the NSE above,

$$-k^2\hat{p}(\mathbf{k}) = -\int\int \delta(\mathbf{k}' + \mathbf{k}'' - \mathbf{k})(-k_i'k_j'')\hat{u}_j(\mathbf{k}')\hat{u}_i(\mathbf{k}''). \quad (17)$$

From substituting $\hat{p}$ in (15), and using the incompressibility condition, $\partial_i u_i = 0 \Rightarrow k_i\hat{u}_i(\mathbf{k}) = 0$, we get the spectral NSE,

$$\partial_t \hat{u}_i(\mathbf{k}) = -ik_j \int \Big(\delta_{il} - \frac{k_i k_l'}{k^2}\Big)\hat{u}_j(\mathbf{k}')u_l(\mathbf{k} - \mathbf{k}')d\mathbf{k}' - \nu k^2\hat{u}_i(\mathbf{k}) + \hat{f}(\mathbf{k}). \quad (18)$$

If we consider the flow in a box of size L^3 with periodic boundary conditions the Fourier transform is substituted by a Fourier series and the integral in (18) becomes a sum,

$$\partial_t \hat{u}_i(\mathbf{n}) = -i(2\pi/L)n_j \sum_{\mathbf{n}'} \Big(\delta_{il} - \frac{n_i n_l'}{n^2}\Big)\hat{u}_j(\mathbf{n}')\hat{u}_l(\mathbf{n}-\mathbf{n}') - \nu n^2\hat{u}_i(\mathbf{n}) + \hat{f}(\mathbf{n}). \quad (19)$$

where the wave vectors are $\mathbf{k}(\mathbf{n}) = 2\pi\mathbf{n}/L$. This form of the NSE is the starting point for the shell models. The partial differential equation (PDE) (1) has now been substituted by a hierarchy of coupled ordinary differential equations (ODE's). The non-linear terms are quadratic in the velocities, so that three waves are coupled. The interactions are such that only waves with wave vectors adding up to zero are coupled. Such a set of three waves is called a triad. It can be shown by manipulating indices in (19) that the inviscid energy conservation fulfilled by the NSE is a detailed energy balance, so that energy is exchanged within each triad. The algebra involved in proving this and many other relations of the NSE is much simpler but completely similar in the case of shell models. We will thus for transparency do many of the calculations in the case of the shell models.

3.1 The spectral energy density

The second order structure function (12) is related to the spectral energy density through a Fourier transform, as we will now show. The energy density of the flow can be expressed in spectral form by use of Parseval's identity,

$$
\begin{aligned}
E = \frac{1}{2}\int \mathbf{u}(\mathbf{x})^2 d\mathbf{x} &= \frac{1}{2}(2\pi)^3 \int \hat{u}_i(\mathbf{k})\hat{u}_i(\mathbf{k})^* d\mathbf{k} \\
&= \frac{1}{2}(2\pi)^3 4\pi \int k^2 |\mathbf{u}(k)|^2 dk \equiv \int E(k)dk,
\end{aligned}
\tag{20}
$$

where we have assumed the flow to be isotropic, $\hat{u}_i(\mathbf{k}) = u_i(k)$ and performed the integration over the sphere. Thus we define the spectral energy density as,

$$
E(k) = (2\pi)^4 k^2 |\mathbf{u}(k)|^2. \tag{21}
$$

The Fourier transform of the velocity is expressed in terms of the second order structure function,

$$
\begin{aligned}
S_2(\mathbf{l}) = \langle \delta\mathbf{u}(\mathbf{l})^2 \rangle = \int (\mathbf{u}(\mathbf{l}+\mathbf{x}) - \mathbf{u}(\mathbf{x}))^2 d\mathbf{x} = \\
2\int [\mathbf{u}(\mathbf{x})^2 - \mathbf{u}(\mathbf{l}+\mathbf{x})\mathbf{u}(\mathbf{x})] d\mathbf{x},
\end{aligned}
\tag{22}
$$

By comparing (20), (21) and (22) and using $\int e^{i\mathbf{kx}} d\mathbf{x} = (2\pi)^3 \delta(\mathbf{k})$ we obtain,

$$
\begin{aligned}
|\mathbf{u}(k)|^2 = \frac{1}{(2\pi)^6} \int\int e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})} u_i(\mathbf{x}) u_i(\mathbf{y}) d\mathbf{x} d\mathbf{y} = \\
\frac{1}{(2\pi)^6} \int e^{i\mathbf{kl}} [\int u_i(\mathbf{l}+\mathbf{y}) u_i(\mathbf{y}) d\mathbf{y}] d\mathbf{l} = \frac{1}{(2\pi)^6} \int e^{i\mathbf{kl}} [\langle \mathbf{u}^2 \rangle - S_2(l)/2] d\mathbf{l} = \\
\frac{1}{(2\pi)^3} \langle \mathbf{u}^2 \rangle \delta(\mathbf{k}) - \frac{1}{(2\pi)^6} \int\int\int e^{ikl\cos\theta} \frac{S_2(l)}{2} l^2 \sin\theta d\phi d\theta dl = \\
\frac{1}{(2\pi)^3} \langle \mathbf{u}^2 \rangle \delta(\mathbf{k}) - \frac{1}{(2\pi)^6} 2\pi \int \frac{1}{2} [-\frac{e^{iklx}}{ikl}]_{x=-1}^{x=1} l^2 S_2(l) dl = \\
\frac{1}{(2\pi)^3} \langle \mathbf{u}^2 \rangle \delta(\mathbf{k}) + \frac{1}{(2\pi)^5} \int \sin kl \Big(\frac{l}{k}\Big) S_2(l) dl,
\end{aligned}
\tag{23}
$$

where we have used homogeneity in the substitution $\mathbf{l} = \mathbf{x} - \mathbf{y}$ and isotropy in performing the integration over the sphere. Finally we get,

$$E(k) = \frac{1}{2\pi} k^{-1} \int_0^\infty x \sin x S_2(x/k) dx, \tag{24}$$

where we have substituted $x = kl$. Inserting the scaling relation (12) for the second order structure function we get,

$$E(k) \sim \bar{\varepsilon}^{2/3} k^{-5/3}. \tag{25}$$

This is the celebrated Kolmogorov scaling of the energy spectrum. This relation could as well be obtained by the same dimensional counting as was used for deriving the scaling relation for the velocity increments. The scaling according to (25) in the inertial range of the energy spectrum for developed 3D turbulence has been demonstrated in many experiments and seen in many observations. Figure 3.1 shows an example from the atmospheric boundary layer where the Reynolds number is of the order 10^5.

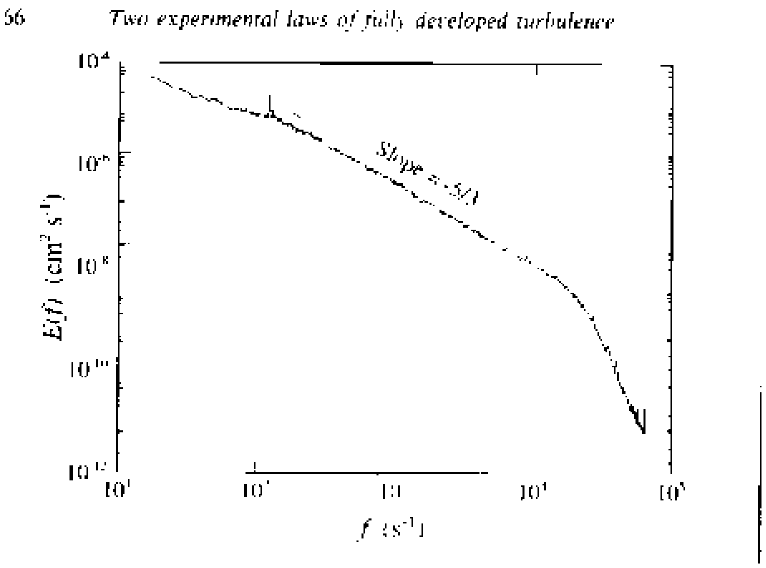

Figure 3: The energy spectrum measured using the Taylor hypothesis in a wind tunnel experiment

3.2 The spectral energy flux

The energy cascade in the Richardson picture is expressed quantitatively through the spectral flux of energy due to the non-linear terms in the NSE. It is defined as,

$$\Pi(k) = \frac{d}{dt}|_{n.l.} \int^k E(k')dk', \tag{26}$$

where the derivative denotes the rate-of-change due to the non-linear terms. From (21) and the spectral NSE (18) the flux can be expressed explicitly in terms of the spectral velocities,

$$\Pi(k) = \int_{|\mathbf{k}'|<k} (2\pi)^4 k^2 \hat{u}_i(\mathbf{k}')^* \frac{d}{dt}|_{n.l.} \hat{u}_i(\mathbf{k}')d\mathbf{k}' =$$

$$\int_{|\mathbf{k}'|<k} (2\pi)^4 k^2 \hat{u}_i(\mathbf{k}')^* [-ik'_j \int \Big(\delta_{il} + \frac{k'_i k''_l}{k'^2}\Big) \hat{u}_j(\mathbf{k}'') u_l(\mathbf{k}' - \mathbf{k}'') d\mathbf{k}''] d\mathbf{k}'. \tag{27}$$

4 Velocity correlations and the closure problem

The statistics of a turbulent flow is characterized by the correlations between the velocity vectors in different point. In tensor notation we can define a two point correlation function as,

$$c_{ij} = \langle u_i(\mathbf{x}) u_j(\mathbf{x}') \rangle, \tag{28}$$

where the bracket denotes an ensemble average. In a homogeneous flow the translational invariance imply that this correlation function only depends on the distance $\mathbf{x} - \mathbf{x}'$ between the two points. Furthermore, in an isotropic flow the rotational invariance imply that the correlation function only depends on the distance $|\mathbf{x} - \mathbf{x}'|$. The Galilean invariance of the Navier-Stokes equation imply that if a constant velocity U_i is added to all velocities in the flow this would again be a solution. However, the correlation function would change to $\langle (u_i + U_i)(u'_j + U_j) \rangle = c_{ij} + U_i U_j + \langle u_i \rangle U_j + U_i \langle u'_j \rangle = c_{ij} + U_i U_j$. The two last terms vanish if the flow is isotropic where any odd order correlation must be zero. We have introduced the short hand notation $u'_i = u_i(\mathbf{x}')$. If in stead we define correlations between velocity differences as,

$$s_{ij} = \langle (u_i - u'_i)(u_j - u'_j) \rangle \tag{29}$$

we obtain an expression which is independent on the mean velocity of the flow. Furthermore, for $l = |\mathbf{x} - \mathbf{x}'| < L$ the influence of eddies of scales larger than l is merely to sweep the flow at the two points in a rigid motion. This sweeping motion would be present in (28) but not in (29). In general we would expect the small scale structure of the flow to be better represented in (29), which are called structure functions, than in the correlation functions (28). The structure functions can trivially be expressed in terms of the correlation functions by expanding the parenthesis,

$$s_{ij} = \langle u_i u_j \rangle + \langle u'_i u'_j \rangle - \langle u_i u'_j \rangle - \langle u'_i u_j \rangle = \frac{2}{3} \langle u^2 \rangle \delta_{ij} - 2c_{ij}, \tag{30}$$

where homogeneity and isotropy has been assumed. In order to obtain an expression for the correlation functions, we observe that in the case of decaying turbulence, the correlation will in general be time dependent and we can express the time dependence using the Navier-Stokes equation,

$$\begin{aligned} \partial_t \langle u_i u'_j \rangle = \langle (\partial_t u_i) u'_j \rangle + \langle u_i \partial_t u'_j \rangle = \\ \langle (-u_k \partial_k u_i - \partial_i p + \nu \partial_{kk} u_i) u'_j \rangle + \langle u_i (-u'_k \partial'_k u'_j - \partial'_j p' + \nu \partial'_{kk} u'_j) \rangle. \end{aligned} \tag{31}$$

The derivative ∂' denotes derivative with respect to $\mathbf{x}'$. This means that we can express the evolution of the second order correlation function in terms of third order correlation functions involving velocities and derivatives of velocities. The pressure can by the incompressibility be expressed by solving the Poisson equation (3) involving a term which is quadratic in the velocities. The dissipation terms are linear and can by isotropy be expressed as $\nu \partial_{kk} \langle u_i u'_j \rangle$. This equation is not closed. In order to close the equation we need equations for the time evolution of the third order correlation functions. These are readily obtained by the same procedure, taking the derivative with respect to time of the third order correlators and applying the Navier-Stokes equation for each term. This will yield an equation involving fourth order correlation functions from the non-linear terms in the Navier-Stokes equation, and thus need a specification for those in order to be closed. We finally end up with an infinite hierarchy of equations which never closes. This is called the closure problem of turbulence. A whole branch of turbulence research deals with development of suitable approximations for closing the equations. The most commonly used closure is to assume Gaussian statistics for the velocities from which the higher order correlation functions can be expressed by the pair correlation functions through Wick's theorem. We shall not stroll further along this path here. A recent account and references for various closures can be found in Lesieurs book.

5 The four-fifth law

A special situation occurs for a third order structure function which can be associated with energy conservation. The spectral flux of energy can be expressed in terms of a third order structure function. This can easily be verified by dimensional counting. The relation was first derived by von Kármán and Howarth [1]. Since energy is conserved by the non-linear terms in the Navier-Stokes equation the mean energy flux must be equal to the mean energy dissipation $\overline{\varepsilon}$. This leads to an exact scaling relation for this third order structure function in the inertial range where energy dissipation can be neglected in comparison to the non-linear flux of energy. This scaling relation was derived by Kolmogorov [3] and is called the four-fifth law. The reason for the name will be apparent shortly. The four-fifth law is derived in great detail in Frisch' book. A completely analogous relation can be derived for the shell models where the algebra is much simpler and transparent. What follows is a short derivation based on Landau and Lifshitz' Fluid Mechanics [5]. Consider a homogeneous and isotropic turbulent fluid, realized by a vigorous stirring and then left alone. The motion of this fluid is governed by the Navier-Stokes equation with no forcing term, so it will eventually have lost all its energy to heat by the viscous dissipation and be at rest. This is called decaying turbulence. The mean energy dissipation is (minus) the mean rate-of-change of kinetic energy, which by interchanging the time derivative and the averaging is given as,

$$\overline{\varepsilon} = -\partial_t \frac{1}{2}\langle u^2 \rangle. \tag{32}$$

In order to utilize the isotropy we will now orient one axis of the coordinate system along the distance vector $\mathbf{l} = \mathbf{x} - \mathbf{x}'$ connecting two points. This coordinate is called the longitudinal direction and denoted by the suffix $_l$ (which should not be confused with the length of the distance vector). The two perpendicular transversal directions are then by isotropy indistinguishable and denoted by the suffix $_t$. From (30) the right hand side can be expressed in terms of the longitudinal components of a two point structure function and a two point correlation function,

$$\bar{\varepsilon} = -\frac{3}{4}\partial_t(s_{ll} + 2c_{ll}). \tag{33}$$

Note that terms like c_{tl} and s_{tl} vanish because of isotropy. The terms c_{tt} and s_{tt} are related to the longitudinal components, but we shall not need that here. From the Navier-Stokes equation we have the expression (31) for the rate-of-change of c_{ij},

$$\partial_t c_{ij} = -\partial_k(c_{ki,j} + c_{kj,i}) + 2\nu\partial_{kk}c_{ij}. \tag{34}$$

where we have defined,

$$c_{ki,j} = \langle u_k u_i u'_j \rangle. \tag{35}$$

To arrive at (34) from (31) requires several steps. Firstly, we can interchange $\mathbf{x} \leftrightarrow \mathbf{x}'$ and simultaneously substitute ∂ for ∂' by a simple rotation using isotropy. Secondly, moving the derivative ∂_k outside the averaging is done using incompressibility; $\partial_k u_k = 0$. Thirdly, the terms involving the pressure vanish: Using again incompressibility we have,

$$\partial_i \langle u_i p' \rangle = 0. \tag{36}$$

The vector $\langle u_i p' \rangle$ can because of isotropy only depend linearly on the unit vector $\hat{l}_i = l_i/l$, $l = |\mathbf{l}|$, in the direction $\mathbf{l} = \mathbf{x}' - \mathbf{x}$. Thus $\langle u_i p' \rangle = f(l)\hat{l}_i$. The general solution to (36) of this form is $\langle u_i p' \rangle = (c/l^2)\hat{l}_i$, with c constant. This is only finite for $l \to 0$ when $c = 0$ and we have $\langle u_i p' \rangle = 0$. To proceed from (34) we will use isotropy to obtain an expression for the third order correlation function $c_{ki,j}$. This is a third order tensor which is symmetric in the two first suffices. The most general form of this tensor, only depending on $\hat{l}_i$, and the unit tensor δ_{ij} is,

$$c_{ki,j} = A(l)\delta_{ik}\hat{l}_j + B(l)(\delta_{kj}\hat{l}_i + \delta_{ij}\hat{l}_k) + C(l)\hat{l}_k\hat{l}_i\hat{l}_j. \tag{37}$$

The prefactor of the two terms in the parenthesis must be the same ($B(l)$) due to the symmetry between k and i. This excludes the antisymmetric tensor of rank three ϵ_{ijk} as well. We can derive relations between the prefactors $A(l), B(l)$ and $C(l)$ by using the incompressibility. Taking the derivative of $c_{ki,j}$ with respect to the second coordinate, we get from (35),

$$\partial'_j c_{ki,j} = \langle u_k u_i \partial'_j u'_j \rangle = 0. \tag{38}$$

Using the expression (37) for the correlation function we get,

$$\begin{aligned} \partial'_j c_{ki,j} &= A' \partial_j l \delta_{ki} \hat{l}_j + A \delta_{ki} \partial_j \hat{l}_j \\ &+ B' \partial_j l (\delta_{kj} \hat{l}_i + \delta_{ij} \hat{l}_k) + B(\delta_{kj} \partial_j \hat{l}_i + \delta_{ij} \partial_j \hat{l}_k) \\ &+ C' \partial_j l \hat{l}_k \hat{l}_i \hat{l}_j + C(\hat{l}_i \hat{l}_j \partial_j \hat{l}_k + \hat{l}_k \hat{l}_j \partial_j \hat{l}_i + \hat{l}_k \hat{l}_i \partial_j \hat{l}_j). \end{aligned} \tag{39}$$

On the right hand side $'$ denotes differentiation with respect to the argument. By straight forward differentiation we have $\partial_j l = \partial_j \sqrt{l_k l_k} = \hat{l}_j$, $\partial_j \hat{l}_i = (\delta_{ij} - \hat{l}_i \hat{l}_j)/l$ and $\partial_j \hat{l}_j = 2/l$, so the expression above becomes,

$$\partial'_j c_{ki,j} = A' \delta_{ki} + 2A\delta_{ki}/l + 2B' \hat{l}_k \hat{l}_i + 2B(\delta_{ki} - \hat{l}_k \hat{l}_i)/l + C' \hat{l}_k \hat{l}_i + 2C \hat{l}_k \hat{l}_i/l. \tag{40}$$

From (38) we get the two equations,

$$\begin{aligned} A' + 2(A+B)/l = 0 \\ (3A' + 2B' + C') + (6A + 4B + 2C)/l = 0, \end{aligned} \tag{41}$$

where the first equation comes from the term proportional to δ_{ki} and the last equation from taking the trace. The last equation can after multiplying by l^2 be rewritten,

$$\begin{aligned} [l^2(3A + 2B + C)]' = 0 \Rightarrow \\ 3A + 2B + C = c/l^2, \end{aligned} \tag{42}$$

where c is a constant of integration. For $l = 0$ the correlation function must vanish as any odd order one point correlation function due to isotropy. Thus we have $c = 0$ and we finally get,

$$B = -A - A'l/2$$
$$C = A'l - A, \tag{43}$$

leaving us with only one prefactor $A(l)$,

$$c_{ki,j} = A\delta_{ki}\hat{l}_j - (A + A'l/2)(\delta_{kj}\hat{l}_i + \delta_{ij}\hat{l}_k) + (A'l - A)\hat{l}_k\hat{l}_i\hat{l}_j. \tag{44}$$

'A' can finally be expressed in terms of the third order structure function,

$$s_{kij} = \langle (u_k - u'_k)(u_i - u'_i)(u_j - u'_j)\rangle = 2(c_{ki,j} + c_{jk,i} + c_{ij,k}). \tag{45}$$

The last identity follows trivially from expanding the parenthesis and using homogeneity and isotropy. From (44) we get $c_{ll,l} = -2A$ and from (45) we get,

$$A = -s_{lll}/12. \tag{46}$$

With this result we can now calculate the first term on the right hand side of (34) which by a trivial but lengthy calculation gives,

$$\partial_k c_{ki,j} = \frac{1}{l}(-2A + 2A'l + A''l^2/2)\hat{l}_i\hat{l}_j + \frac{1}{l}(-2A - 3A'l - A''l^2/2)\delta_{ij} \tag{47}$$

With this and combining (33), the purely longitudinal component of (34) and (46) we get,

$$-\frac{2}{3}\bar{\varepsilon} - \frac{1}{2}\partial_t s_{ll} = \frac{1}{6l}(4s_{lll} + l s'_{lll}) + 2\nu\partial_{kk}(\frac{1}{3}\langle v^2\rangle - \frac{1}{2}s_{ll}). \tag{48}$$

The second term on the left hand side must be small in comparison to the first on scales $l \ll L$ since in a decaying turbulence the time scale over which the flow can be regarded as (semi-)steady state is of the order of the large scale turnover time L/U so this term can be neglected. The viscous term involving $\partial_{kk}\langle u^2\rangle$ vanishes due to incompressibility. In the inertial range $l \gg \eta$ we can neglect the viscous term all together. The first term on the right hand side can be rewritten as $(l^4 s_{lll})'/(6l^4)$, so with rearranging the terms we get,

$$-\frac{4}{5}(\bar{\varepsilon}l^5)' = (l^4 s_{lll})' \Rightarrow s_{lll} = -\frac{4}{5}\bar{\varepsilon}l. \tag{49}$$

This is the four-fifth law which is a cornerstone in fluid mechanics. The scaling relation is in agreement with the phenomenology described in the K41 theory, but the four-fifth law is, in contrast to the K41 phenomenology, exact in the proper limit and among the few exact results derived from the Navier-Stokes equation.

6 The dissipative anomaly

The four-fifth law was derived using energy conservation and the NSE. If we again consider the evolution of the energy and integrate over the volume of the flow,

$$\frac{1}{2}\partial_t \int u^2 d\mathbf{x} = \nu \int u_i \partial_{jj} u_i d\mathbf{x} + \int u_i f_i d\mathbf{x} = -\nu \int [(\partial_j u_i)^2 + F], d\mathbf{x} \qquad (50)$$

where F is the rate of energy pumped into the flow at large scales. The integral of the non-linear terms in the NSE involves by use of incompressibility only integrals of divergences leaving only boundary terms which vanish if the flow is at rest at the boundaries or if we impose periodic boundary conditions. In a steady state the time mean of two terms on the right hand side must balance. Now in the case of a vanishing viscosity, $\nu \to 0$, the energy dissipation must remain constant and consequently the integral,

$$\int (\partial_j u_i)^2 d\mathbf{x} = \int \omega^2 d\mathbf{x}, \qquad (51)$$

must diverge. Here we have introduced the vorticity $\omega_i = \epsilon_{ijk}\partial_j u_k$. This means that velocity gradients in the flow are unlimited in the limit $\mathsf{Re} \to \infty$. This is called the dissipative anomaly. In order for the dissipation to remain constant the integral of the squared vorticity, which we will later denote the enstrophy, diverges. This again implies that the energy cascade to the small scales must be accompanied by an efficient production of squared vorticity. By applying the operator $\epsilon_{ijk}\partial_j$ to the NSE we obtain the vorticity equation,

$$\partial_t \omega_i + u_j \partial_j \omega_i = -\omega_j \partial_j u_i + \nu \partial_{jj} \omega_i + \epsilon_{ijk} \partial_j F_k, \qquad (52)$$

where the pressure term vanishes identically and the first term on the right hand side appears from the advection term in the NSE by a lengthy but simple tensor manipulation. The left hand side is just the material derivative of the vorticity while the two last terms on the left hand side are dissipation and forcing. The first term on left hand side is the non-linear term responsible for the production of vorticity at the small scales. It is called the stretching and bending term which can be split into two terms representing vortex stretching and bending. The derivation of this is a straight forward algebra but too much of a digression from the lines of thought presented here. The first term signifies the creation of vorticity by stretching vortices along the axis of rotation. This is the 'ballet dancer' effect of increasing spin by pulling in the arms. The second term is the creation of vorticity from the bending of vortices caused by velocity shear along the axis of rotation. This is observed in the winding of tornadoes. From the dissipative anomaly and the vorticity equation we expect the energy dissipation and small scale vortices to be intimately linked in fully developed turbulence.

7 Intermittency

The four-fifth law establishes an exact scaling relation for a third order structure function. For other orders the K41 scaling theory predicts the scaling,

$$S_p(l) = \langle \delta u(l)^p \rangle \sim l^{p/3}, \tag{53}$$

which is the only one that can be obtained by dimensional counting. This would imply that $\langle \delta u(l)^{2q} \rangle \sim \langle \delta u(l) \rangle^{2q}$ as is the case if $\delta u(l)$ was Gaussian. By a field being Gaussian we mean that it has a Gaussian distribution. If the field is not Gaussian, we should not expect (53) to hold in general. From the four-fifth law we know that $\delta u(l)_{||}$ must have a skewed distribution, since the odd ordered structure functions would vanish for a symmetric distribution. High Reynolds number flow is observed to behave strongly intermittent. Calm periods are interrupted by sudden bursts of energy at small scales which are then effectively dissipated. There is now conclusive experimental evidence that corrections to the scaling law (53) are present in the inertial range of fully developed turbulence. The structure functions still scale with length, but the scaling exponents $\zeta(p)$ defined from,

$$S_p(l) \sim l^{\zeta(p)}, \tag{54}$$

are different from $p/3$. Figure 7 shows the scaling exponent $\zeta(p)$ as a function of p for turbulence in a wind tunnel [6].

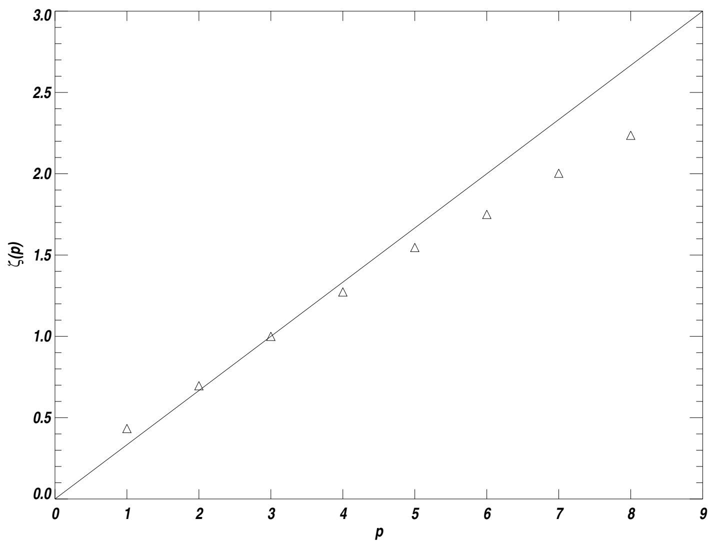

Figure 4: The anomalous scaling exponents derived from the structure functions measured in a wind tunnel experiment.

The straight line is the K41 prediction. The function $\zeta(p)$ is called the anomalous scaling exponents, 'anomalous' referring to the inhomogeneous nature of the intermittent flow. The bursts of strong energy dissipation are associated with small scale structures of high vorticity. Whether these structures are vortex sheets, vortex tubes or more complicated filament structures is not completely well understood. A phenomenological description of the connection between these structures and the anomalous scaling is given by the fractal- or multiplicative models which we shall describe briefly later. Calculating the anomalous scaling exponents from the NSE is to date a major challenge at which there has been little success despite a rather large effort in recent years. Shell models show intermittency corrections to the K41-like scaling as well and for shell models it is relatively easy, at least qualitatively, to understand how the intermittency emerges. Even though shell models cannot account for the spatial structures in fully developed turbulent flow the intermittent behavior of these toy-models has been a major motivation for studying them.

7.1 Finite time singularities

The dissipative anomaly implies that if the flow is initially a smooth field the velocity gradients will grow in time. It is today an open question whether the gradients will blow up in finite time or if there are solutions to the Navier-Stokes equation for arbitrarily high Reynolds numbers which will stay smooth for all times. If the solutions to the NSE are irregular, we must define the differential operators in terms of distributions. Such singular solutions $u_i(\mathbf{x}, t)$ satisfy the integral equation,

$$\int (u_i \partial_t + u_i u_j \partial_j - p\partial_i + \nu u_i \partial_{jj} + f_i)\phi_i d\mathbf{x}dt = 0 \tag{55}$$

where $\phi_i(\mathbf{x}, t)$ is any smooth function with compact support, a so called test function. This equation is derived from the NSE by, formally, moving all the differentiations of the velocity field $u_i(\mathbf{x}, t)$ by integration by part to the smooth test field. Correspondingly we have

$$\int u_i \partial_i \psi d\mathbf{x}dt = 0 \tag{56}$$

for the incompressibility. Here $\psi(\mathbf{x}, t)$ is any scalar test function. Such solutions are called weak solutions of the NSE [7]. A naive argument for a finite time blow-up in the NSE equation can be given in terms of eddy turnover times. The eddy turnover time τ_l at scale l is from dimensional counting proportional to $\overline{\epsilon}^{-1/3} l^{2/3}$, taking the time for an energy burst to be transferred from a scale l to $l/2$ to be proportional to τ_l. This gives a convergent quotient sum and thus a finite time for the energy transfer to infinitely small scales. The question of finite time blow-up in the NSE has been posed as one of only seven 'Hilbert problems' for the twenty-first century, with an award for it's solution of 1 million dollars by the Clay Math foundation.

8 2D turbulence

If the flow is constrained to two dimensions, as is the case for some stratified fluids and to a good approximation the free atmosphere on Earth, the flow will be fundamentally different from the flow in three dimensions. If we consider the vorticity equation (52) it contains the stretching and bending term which ensures the production of enstrophy necessary for the dissipation to remain constant in the limit of vanishing viscosity. This term will trivially vanish in two dimensions since the vorticity is perpendicular to the plane of the flow. In 2D the vorticity can be defined as a scalar $\zeta = \epsilon_{ij}\partial_i u_j$, where $\epsilon_{11} = \epsilon_{22} = 0$ and $\epsilon_{12} = -\epsilon_{21} = 1$, and the vorticity equation reduces to the scalar equation,

$$D_t\zeta = \partial_t\zeta + u_j\partial_j\zeta = -\nu\partial_{jj}\zeta + \epsilon_{ij}\partial_i f_j. \tag{57}$$

This means that the vorticity acts as a passively advected quantity and the mean square vorticity, the enstrophy, is an inviscid invariant of the flow along with the energy. If we consider an unforced flow at high Reynolds number, then the vorticity must grow in order to dissipate the energy at small scales. This is, however, not possible since the vorticity can only decrease from its initial value due to dissipation of vorticity. Thus a cascade of energy to small scales is impossible in 2D turbulence. What happens then? The energy and enstrophy equations are,

$$D_t \int \frac{u^2}{2} d\mathbf{x} = -\nu \int \omega^2 d\mathbf{x} \tag{58}$$

and

$$D_t \int \frac{\omega^2}{2} d\mathbf{x} = -\nu \int (\partial_i\omega_i)^2 d\mathbf{x}. \tag{59}$$

These equations can readily be written in the spectral representation,

$$D_t \int E_{\mathbf{k}} d\mathbf{k} = -\nu \int k^2 E_{\mathbf{k}} d\mathbf{k}, \tag{60}$$

and

$$D_t \int Z_{\mathbf{k}} d\mathbf{k} = D_t \int k^2 E_{\mathbf{k}} d\mathbf{k} = -\nu \int k^4 E_{\mathbf{k}} d\mathbf{k}, \tag{61}$$

where $Z_{\mathbf{k}} = \omega_{\mathbf{k}}\omega_{-\mathbf{k}} = k^2 E_{\mathbf{k}}$ is the spectral enstrophy density. The ratio of dissipation of energy to enstrophy at wave number $\mathbf{k}$ is $D_E(\mathbf{k})/D_Z(\mathbf{k}) = k^{-2}$.

Since the dissipation takes place at the Kolmogorov scale $\eta = k_E^{-1}$ the energy dissipation can be estimated as $\overline{z}\eta^2$ where $\overline{z}$ is the mean enstrophy dissipation. Thus the energy dissipation vanishes for high Reynolds number flow. This led Kraichnan [8] to propose a completely different scenario from the 3D case for cascade in 2D turbulence. The conjecture is that enstrophy and not energy is cascaded to the small scales, while the energy on the contrary is cascaded up scale to the large scales. So in 2D turbulence there are two inertial ranges, one for the forward cascade of enstrophy and one for the inverse cascade of energy. In the inertial range for enstrophy cascade we can repeat the dimensional scaling arguments from K41 for the enstrophy. The mean dissipation of enstrophy has the dimension $[\overline{z}] = time^{-3}$. Thus if the enstrophy spectrum $Z(k) = \int_{|\mathbf{k}|=k} Z_{\mathbf{k}} d\Omega$ depends solely on the mean enstrophy dissipation $\overline{z}$ and the scale k^{-1} we get from dimensional counting, $Z(k) \sim \overline{z}^{2/3}k^{-1}$ and using $Z(k) = k^2E(k)$ we obtain the enstrophy cascade energy spectrum,

$$E(k) \sim \overline{z}k^{-3}, \tag{62}$$

for the forward cascade range. For the inverse energy cascade we should expect the K41 scaling (25), since the scaling arguments leading to this scaling nowhere assumes a specified direction of the cascade. The arguments above leading to the forward enstrophy cascade and inverse energy cascade was based on the dissipation of enstrophy inhibiting the dissipation of energy at the small scales. Another qualitative argument involving only the non-linear terms and the inertial flow goes as follows. Consider an experiment where a 2D fluid is excited solely around a wavenumber k_0, suppressing vector notation in the following. The triad interactions will then act as to diffuse the energy at wavenumber k_0 locally under the constraint that the two integrals,

$$E = \int E(k)dk, \;\; Z = \int k^2 E(k)dk \tag{63}$$

are constant. Since the large k tail will dominate the second integral, this can only happen if the maximum of the curve moves towards smaller wave numbers and up scale energy transfer accompany down scale enstrophy transfer. This is schematically shown in figure 8.

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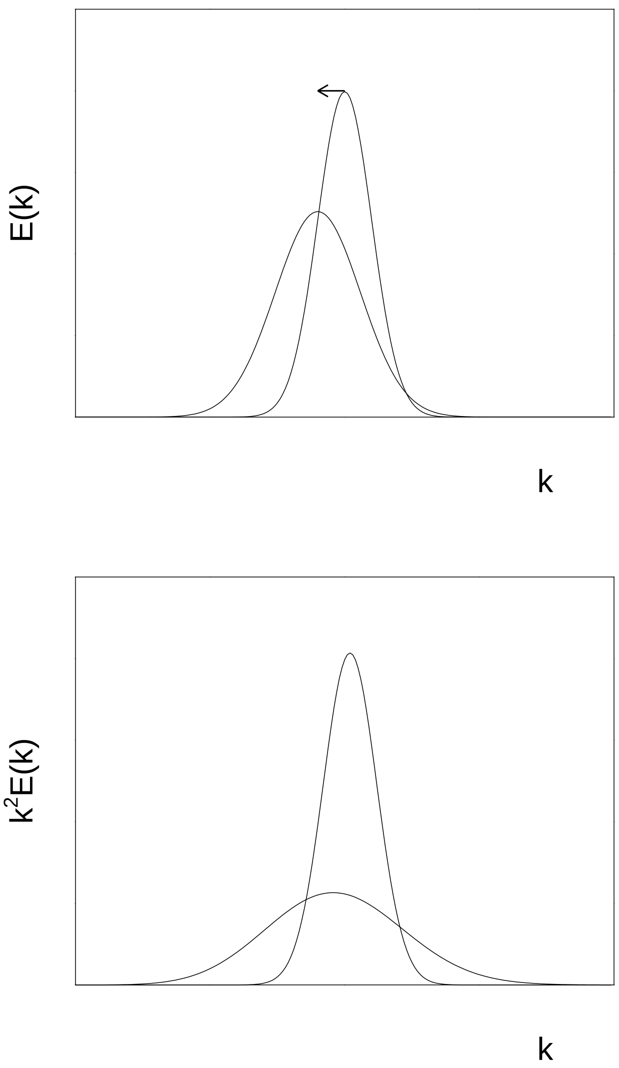

Figure 5: The energy initially concentrated around wave number k_0 will spread to neighboring wave numbers through the non-linear wave interactions. In order to conserve both energy and enstrophy the energy is cascaded up scale and the enstrophy is cascaded down scale simultaneously.