Symmetries, invariants, and cascades in a shell model of turbulence

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Reduced wave number models of turbulence, namely shell models, show cascade processes and anomalous scaling of correlators which might be analogous to what is observed in Navier-Stokes (NS) turbulence. The scaling properties of the shell models depend on the specific symmetries and invariants of the models. A shell model is investigated. It is argued that this model might have a closer resemblance than the standard Gledzer-Ohkitani-Yamada model to the NS turbulence. The shell model investigated here coincides with the Sabra model proposed by L'vov *et al.* [Phys. Rev. E. **58**, 1811 (1998)] for a specific choice of the free parameters of their model. For this choice of parameters, besides the energy and the ''helicity,'' the model has a cubic inviscid invariant.

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I. INTRODUCTION

The connection between Kolmogorovs 1941 (K41) scaling theory of turbulence and the Navier-Stokes (NS) equation is through the four-fifth law [1]. The four-fifth law is one of the few rigorous result regarding three-dimensional (3D) hydrodynamic turbulence, connecting the second- and third-order longitudinal velocity structure functions. The equation is not closed, but in the inertial range, $\eta \ll r \ll L$, η being the inner (Kolmogorov) scale and L the outer (integral) scale, we have $S_3(r) = -(4/5)\epsilon r$, where ϵ is the mean energy dissipation. This implies $\zeta(3)=1$, where $\zeta(p)$ is defined from the scaling of the longitudinal structure functions, $S_p(r) = \langle \delta v_\parallel(r)^p \rangle \sim r^{\zeta(p)}$. The scaling of all other structure functions is observed to deviate from the K41 prediction, which from dimensional counting would imply $\zeta(p)=p/3$. The deviation from the K41 prediction is called anomalous scaling exponents, referring to the intermittent nature of the energy dissipation, where the energy is inhomogeneously dissipated.

As was noted recently [2], the inviscid conservation of helicity leads to another exact scaling law for a third-order correlator associated with the flux of helicity, provided an inertial range for helicity flux exists [3]. Similarly for any inviscid invariant an exact scaling law can be derived for the correlators associated with the spectral flux of such a quantity provided there is an inertial range separating sources and sinks for this quantity. This is the case for the shell model investigated here, where the existence of a third-order inviscid invariant leads to a scaling law for a fourth-order correlator. This has been argued to be a calculation of an anomalous scaling exponent [4]. This is, however, not an anomalous scaling exponent in the aforementioned sense; it would still be present in the case of an (imagined) homogeneous nonintermittent flow. On the contrary, the scaling of this specific fourth-order correlator will be ''normal'' in the same sense as $\zeta(3)=1i$ from the four-fifth law.

The recent interest in shell models is mainly that they show numerically the same type of intermittent behavior as seen in 3D turbulence [5]. The simplicity of shell models makes it possible to calculate the anomalous scaling exponents with high accuracy, which is still an undoable task for the NS equation. So in this sense shell models might prove useful as the starting point for exact results regarding calculating scaling exponents. The introduction of the Sabra model [6] as superior to the Gledzer-Ohkitani-Yamada (GOY) model [7,11] was motivated by improvements in accuracy with respects to numerical determination of scaling exponents. Here the emphasis will be on the similarity with the NS equation and the cascade properties of the third-order invariant. In fact, the model will not exhibit a cascade of the third-order quantity, which is easily seen from a scaling argument and confirmed in a numerical simulation. Thus it will be shown that the scaling exponent obtained in [4] is irrelevant.

II. THE GOY MODEL

The GOY model has built in the K41 scaling in the sense that the K41 scaling is a fixed point of the model. Furthermore, it has an unfortunate modulus (3) symmetry in shell numbers which has no resemblance in the NS equation and which makes a precise numerical determination of scaling exponents difficult [8,4]. The modulus (3) symmetry, furthermore, introduces artificial long-range (in k space) correlations with no analogs in the NS equation. The GOY model has two inviscid invariants, the energy and a second nonpositive definite quantity dimensionally equivalent to the helicity in 3D NS turbulence. This ''helicity'' only vaguely resembles the helicity in the NS fluid, and it has been argued that it leads to an anomalous scaling behavior of the GOY model different from the mechanisms for intermittency in NS turbulence [9]. A review of the main differences between the GOY model and the Sabra model is given in Ref. [6].

III. THE SHELL MODEL

The shell model, defined in the following, can be regarded as a special case of the Sabra model introduced by L'vov *et al.* [6]. It has the same two quadratic inviscid invariants, energy and ''helicity'' as the GOY model. Furthermore, it has one cubic inviscid invariant. The energy is the only positive invariant. As for the NS equation, and in contrast to the GOY model, the K41 scaling is not a fixed point of the model.

The model is defined in the usual way by a set of exponentially spaced one-dimensional wave numbers $k_n = k_0\lambda^n$, for which we have u_n as the complex shell velocity for shell n $(n=1,N)$. The form of the governing equation for the model is motivated by two demands. First, the momenta involved in the triad interactions must add up to zero as in the NS equation. Second, the complex conjugations involved in the nonlinear term should be the same as for the terms in the spectral NS equation involving triads, for which the moduli of the wave vectors fall within three consecutive shells. This together with the usual construction of local interactions in k space, inviscid conservation of energy, and fulfillment of Liouville's theorem gives the equation of motion for the shell velocities,

$$(d/dt+\nu k_n^2)u_n = i[k_n u_{n+1}^* u_{n+2} - \epsilon k_{n-1} u_{n-1}^* u_{n+1} + (1-\epsilon)k_{n-2}u_{n-1}u_{n-2}] + f_n, \qquad (1)$$

where ν is the viscosity and f_n is the external forcing. The forcing would as in the GOY model typically be taken to be active only for some small wave numbers, e.g., $f_n = f\delta_{n,4}$.

Boundary conditions can be specified in the usual way by the assignment $u_{-1}=u_0=u_{N+1}=u_{N+2}=0$.

The first requirement is fulfilled if the wave numbers k_n are defined as a Fibonacci sequence, $k_n = k_{n-1}+k_{n-2}$. The choice of a Fibonacci sequence for the momenta leads to a model with the shell spacing uniquely being the golden ratio g, since for any choice of $k_1, k_2 (k_1 \leqslant k_2)$ we have $k_n/k_{n-1} \to g$ for $n\to\infty$. So this corresponds to the usual definitions of the shell wave numbers with the golden ratio as shell spacing for $k_1=1, k_2=(\sqrt{5}+1)/2 \equiv g$ [10]. The golden ratio $g=(1+\sqrt{5})/2$ plays a key role in the symmetries of shell models.

With this formulation, the shell spacing is not a free parameter of the shell model. However, using the definition by L'vov *et al.* of $k_n = g^n$ being a "quasimomentum," we shall keep the shell spacing, λ, as a free parameter, $k_n=\lambda^n$.

If we interpret the momenta, k_n, as representative of the modulus of the wave vector, $\mathbf{k}_n$, in 2D or 3D, the triangle inequality implies $k_n + k_{n+1} \geqslant k_{n+2}$ so the Fibonacci sequence corresponds in this sense to moduli of three parallel wave vectors. Note that for a shell spacing $\lambda > g$ (as the usual choice $\lambda=2$), the triangle inequality is violated. This means that we cannot interpret the usual shell-model interactions as representative interactions between waves within three consecutive shells, since no such triplets of wave numbers constitute triangles.

In order to give meaning to the notion of closing the triads, we define negative momenta, $k_{-n} \equiv -k_n$, and assign the velocity, $u_{-n}=u_n^*$, to these momenta. (The model still only has $2N$ degrees of freedom, represented by the N complex velocities.) Note that Eq. (1) is also fulfilled for the negative momenta, which is why the prefactor must be "i." With this notation we can rewrite Eq. (1) as

$$(d/dt+\nu k_n^2+)u_n = ik_n \sum_{k_l<k_m} \tilde{I}(l,m;n)u_l u_m + f_n, \qquad (2)$$

where the sum is over positive and negative momenta, and all the dimensionless interaction coefficients have the simple form $\tilde{I}(l,m;n) = I(l,m;n)\delta_{k_n+k_l+k_m,0}$ with $I(l,m;n) = \delta_{n-2,l}\delta_{n-1,m} - (\epsilon/\lambda)\delta_{n-1,l}\delta_{n+1,m} + [(1-\epsilon)/\lambda]\delta_{n+1,l}\delta_{n+2,m}$. From this formulation it is clear why the complex conjugations in this model are exactly as in Eq. (1). They arise from the closing of the triads in the same way as for the spectral NS equation. As noted in [6], this is the main difference from the GOY model.

IV. INVISCID INVARIANTS

It can easily be shown that there are only second-order (quadratic) invariants of the form $\Sigma \xi^n u_n u_{-n}$. The inviscid conservation of these quadratic invariants is obtained from

$$\frac{d}{dt}\frac{1}{2}\sum_{|n|\leqslant N} \xi^n u_n u_{-n} = i\sum_{k_l+k_m=k_n} \xi^n k_n I(l,m;n) u_l u_m u_{-n} = i \sum_{3\leqslant |n| \leqslant N} \xi^{n-2}k_{n-2}[1-\epsilon\xi-(1-\epsilon)\xi^2] \times u_{n-2}u_{n-1}u_{-n} = 0, \qquad (3)$$

so exactly as for the GOY model we obtain the equation

$$1-\epsilon\xi-(1-\epsilon)\xi^2 = 0 \qquad (4)$$

with the two solutions $\xi=1$ and $\xi=1/(\epsilon-1)$. The first corresponds to energy conservation, with

$$E = \sum E_n = \frac{1}{2}\sum u_n u_{-n}, \qquad (5)$$

and the second to "helicity" conservation (for $\epsilon<1$), with

$$H = \sum H_n = \frac{1}{2}\sum_{|n|\leqslant N}\left(\frac{1}{(\epsilon-1)}\right)^n u_n u_{-n}. \qquad (6)$$

With the definition of negative momenta the only possible pth-order invariants with terms $u_{i_1}\cdots u_{i_p}$ must have the associated momenta summing to zero. Thus for any invariant,

$$\sum_i \xi_i^1 u_{j_1}\cdots u_{j_p} + \cdots + \xi_i^l u_{l_1}\cdots u_{l_p} + \text{c.c.}, \qquad (7)$$

the corresponding momentum vectors must add up to zero, $k_{j_1}+\cdots+k_{j_p} = k_{l_1}+\cdots+k_{l_p}=0$. This can easily be seen by differentiating (7) with respect to time using Eq (2). The only possible third-order term is

$$G = \sum_n G_n = \sum_n \xi^n (u_{n-1}u_n u_{-(n+1)} + \text{c.c.}). \qquad (8)$$

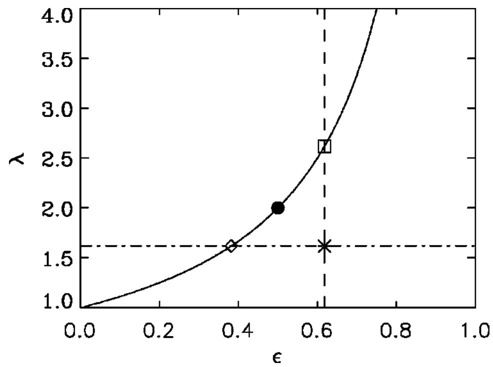

FIG. 1. The parameter space (ϵ,λ) for the shell model. Energy is always conserved by construction. The hyperbola corresponds to conservation of dimensionally correct helicity $H=\Sigma_n (-1)^n k_n |u_n|^2$. The horizontal line is the golden ratio shell spacing where the "triangles" close, $k_{n-2}+k_{n-1}=k_n$. The vertical line represents parameters for which $G=\Sigma_n(-\lambda/\epsilon)^n \times \mathcal{R}[u_{n-1}u_n u_{-(n+1)}]$ is an inviscid invariant. At the cross, "helicity" has the form $H=\Sigma_n(-1)^n k_n^{1/2}|u_n|^2$.

By taking the derivative with respect to time using Eq. (2), we obtain a set of equations similar to Eq. (4):

$$\epsilon\xi+\lambda=0,$$

$$(1-\epsilon)\xi^2-\lambda^2=0, \tag{9}$$

$$(1-\epsilon)\xi+\epsilon\lambda=0.$$

These equations are fulfilled if $1-\epsilon-\epsilon^2=0$ with $\xi=-\lambda/\epsilon$. The two solutions are $\epsilon=-g$ and $\epsilon=1/g$.

From a similar analysis it can also be seen that there are no invariants with $p>3$.

V. PARAMETER SPACE

With ϵ and $\lambda>1$ being the two free parameters of the model, the parameter space is represented in Fig. 1. The hyperbola is the curve $\lambda=1/|\epsilon-1|$ corresponding to the dimensionally correct helicity, $H=\Sigma(-1)^n k_n|u_n|^2$. The dashed vertical lines correspond to values where the third-order quantity G is an inviscid invariant. The point $(\epsilon,\lambda)=(1/g,g^2)$, marked with a square, has both the usual helicity and $G=\Sigma(-1)^n g^{3n} u_{n-1}u_n u_{-(n+1)}$ conserved. The point $(\epsilon,\lambda)=(2-g,g)$, marked with a diamond, does not have G conserved, while the point $(\epsilon,\lambda)=(1/g,g)$, marked with a cross, has G conserved with $\xi=-g^2$ and a helicity of the form $H=\Sigma(-1)^n k_n^{1/2}|u_n|^2$. The point $(\epsilon,\lambda)=(1/2,2)$, marked with a solid ball, is the point investigated by L'vov *et al.* corresponding to the values originally chosen for the GOY model. In the rest of this paper we will investigate the case $(\epsilon,\lambda)=(1/g,g^2)$ in order to discuss the role of the cubic invariant G.

VI. HAMILTONIAN STRUCTURE

The Hamiltonian structure of the model as reported in Ref. [4] can be observed, by change of variables, $v_n=(-\epsilon)^{-n/2}u_n$,

$$\dot{v}_n=-i\frac{\delta[\lambda G/(-\epsilon)]}{\delta v_{-n}}, \tag{10}$$

where $G(v_m,v_{-m})$ is defined as a Hamiltonian with a density $G_n=(\lambda\sqrt{-\epsilon})^n v_{n-1}v_n v_{-(n+1)}$, which is local in wavenumber space. This relation is more a curiosity than of practical importance. For shell models it would be more natural to have a Hamiltonian associated with the energy as in most conservative dynamical systems. The attempt to construct this has, however, not been fruitful until now [8].

VII. THE NONLINEAR FLUXES

The nonlinear transfers of the invariants are defined as the currents $\tilde{\Pi}_n^E=(d/dt)\Sigma_{m\leq n}E_m$ $(\nu=f=0)$ and correspondingly for H. They are

$$\tilde{\Pi}_n^E=k_n[D_{n+1}+(1-\epsilon)D_n/\lambda],$$
$$\tilde{\Pi}_n^H=k_n(\epsilon-1)^{-n}(D_{n+1}-D_n/\lambda), \tag{11}$$

where $D_n=2\,\mathrm{Im}(u_{-(n-1)}u_{-n}u_{n+1})$. For $\epsilon=1/g$ the flux of G is

$$\tilde{\Pi}_n^G=(-\lambda^2/\epsilon)^n(\lambda D_{n+1}^{(1)}-\epsilon D_n^{(1)}/\lambda-\epsilon D_n^{(2)}+D_n^{(3)}), \tag{12}$$

where

$$D_n^{(1)}=2\,\mathrm{Im}(u_{n+2}u_{-(n+1)}u_{-(n-1)}u_{-(n-2)}),$$
$$D_n^{(2)}=2\,\mathrm{Im}(u_{n+2}u_{-n}^2u_{-(n-1)}), \tag{13}$$
$$D_n^{(3)}=2\,\mathrm{Im}(u_{n+2}u_{-(n+1)}^2u_{n-1}).$$

VIII. STRUCTURE FUNCTIONS

Scaling exponents are obtained from expressing the transfers in terms of structure functions. As found in [6] Eq. (1) is invariant under the rotation, $u_n\rightarrow\exp(i\theta_n)u_n$, where the phases are a Fibonacci sequence, $\theta_{n-2}+\theta_{n-1}=\theta_n$. This symmetry is a trivial consequence of the construction of the model. The implication of the symmetry on the structure functions is that $\langle u_{j_1}\cdots u_{j_p}\rangle=\exp[i(\theta_{j_1}+\cdots+\theta_{j_p})]\times\langle u_{j_1}\cdots u_{j_p}\rangle$. Thus only structure functions with $\theta_{j_1}+\cdots+\theta_{j_p}=0$ can be nonzero. Since the phases fulfill the same relations as the associated momenta, we can conclude that only structure functions where the associated momenta sum to zero are nonzero. The corresponding symmetry in the GOY model is $\theta_{n-2}+\theta_{n-1}+\theta_n=0$ leading to the artificial slowly decaying modulus (3) correlation among distant shells. This is argued in [6] to make this model superior to the GOY model.

The nonvanishing structure functions can easily be listed, thus we have the following second-, third- and fourth-order structure functions:

$$S_2(n)=\langle u_n u_{-n}\rangle=\langle E_n\rangle, \tag{14}$$

$$S_3(n)=2\,\mathrm{Im}\langle u_{n-1}u_n u_{-(n+1)}\rangle=\langle D_n\rangle, \tag{15}$$

$$S_4^{(0)}(n,m)=\langle|u_n|^2|u_m|^2\rangle=\langle E_nE_m\rangle, \quad (16)$$

$$S_4^{(1)}(n)=2\,\mathrm{Im}\langle u_{n-2}u_{n-1}u_{n+1}u_{-(n+2)}\rangle=-\langle D_n^{(1)}\rangle, \quad (17)$$

$$S_4^{(2)}(n)=2\,\mathrm{Im}\langle u_{n-2}u_{-n}^2u_{n+1}\rangle=-\langle D_n^{(2)}\rangle, \quad (18)$$

$$S_4^{(3)}(n)=2\,\mathrm{Im}\langle u_{n-2}u_{n-1}^2u_{-(n+1)}\rangle=-\langle D_n^{(3)}\rangle. \quad (19)$$

IX. EXACT SCALING RELATIONS

The exact scaling relations corresponding to the four-fifth law [12] simply state that the mean nonlinear transfers $\Pi_n^{E,H,G}=\langle\tilde{\Pi}_n^{E,H,G}\rangle$ are independent of wave number within the respective inertial ranges in the high Reynolds number limit, $\Pi_n^E=\overline{\epsilon}$, $\Pi_n^H=\overline{\delta}$, and $\Pi_n^G=\overline{\eta}$, where $\overline{\epsilon},\overline{\delta},\overline{\eta}$ are the mean dissipations of E,H,G, respectively. This is usually expressed in terms of structure functions. From Eqs. (11) and (15), we readily obtain

$$k_nS_3(n+1)+k_{n-2}S_3(n)=\overline{\epsilon}, \quad (20)$$

$$k_nS_3(n+1)-k_{n-1}S_3(n)=(-1)^nk_n^{-1}\overline{\delta} \quad (21)$$

in the inertial range, with the solution

$$S_3(n)=\frac{\lambda^2}{k_n(1+\lambda)}[\overline{\epsilon}+(-1)^nk_n^{-1}\overline{\delta}]. \quad (22)$$

The second term on the right-hand side is an oscillatory term which is subleading in the scaling with k_n in comparison to the first term. This term disappears in the case of a ''helicity-free'' forcing where $\overline{\delta}=0$ and we recover the shell-model correspondence to the $\frac{4}{5}$th law expressed in terms of the third order structure function, $S_3(n)=\overline{\epsilon}/k_n$.

The equation similar to Eq. (20) for the transfer of the third-order quantity, G, reads

$$\begin{aligned}k_n^2(-\epsilon)^n[\lambda S_4^{(1)}(n+1)+\epsilon S_4^{(1)}(n)/\lambda-\epsilon S_4^{(2)}(n)+S_4^{(3)}(n)]\\ =k_n^{2+\alpha}F_4(n)\\ =\overline{\eta},\end{aligned} \quad (23)$$

where $\overline{\eta}$ is the mean dissipation of G, $F_4(n)$ denotes the square bracket on the left-hand side, and $\alpha=\ln(-\epsilon)/\ln(\lambda)=[i\pi+\ln(\epsilon)]/\ln(\lambda)$. This was argued by L'vov *et al.* [4] to establish a nontrivial calculation of a (subleading) scaling exponent, which in this notation reads $F_4(n)\sim k_n^{-2-\alpha}=k_n^{\tilde{\zeta}(4)}\Rightarrow\tilde{\zeta}(4)=2+[i\pi+\ln(\epsilon)]/\ln(\lambda)$. The imaginary part of the scaling exponent comes from the $(-1)^n$ factor, which can be trivially eliminated by reformulating the model in terms of new variables, $v_{2n}\equiv u_{2n}$ and $v_{2n+1}\equiv-u_{2n+1}^*$, as was done by L'vov *et al.* [4].

X. INERTIAL RANGES

The validity of the exact scaling relations depends on the existence of inertial ranges separating the sources and sinks for the inviscid invariants solely associated with the nonlinear fluxes of the invariants. The inertial range for the energy flux is determined by the Reynolds number Re, which here we just associate with the inverse of the viscosity ν, assuming velocity at the integral scale to be of order unity. The Kolmogorov scale is in complete analogy to the K41 theory determined by balancing dissipation and nonlinear flux, so we have $K_E\sim(\overline{\varepsilon}/\nu^3)^{1/4}$ growing as $Re^{3/4}$. A similar analysis can be done for the analogous Kolmogorov scale for dissipation of helicity [3]. Balancing dissipation and helicity input using $u_n\sim(\overline{\varepsilon}/k_n)^{1/3}$ gives $K_H\sim[\overline{\delta}^3/(\nu^3\overline{\varepsilon}^2)]^{1/7}$. Thus we get $K_H/K_E\sim\nu^{-3/7+3/4}=\nu^{9/28}\rightarrow0$ for $\nu\rightarrow0$. This means that for high Reynolds number flow the small scales will always exhibit nonhelical flow. In the shell model the helicity changes sign due to the $(-1)^n$ factor. The dissipation of positive helicity at even-numbered shells and negative helicity at odd-numbered shells will grow with the wave number as $D_n^H\sim(-1)^nk_n^3|u_n|^2\sim(-1)^nk_n^{7/3}$, consequently the shell model will show strong odd-even oscillations of Π_n^H from balancing the positive and negative dissipations. The scaling will be determined by the dissipation $|\Pi_n^H|\sim k_n^{7/3}$ for $k_n>K_H$. The situation for the cubic invariant G is different. We can again define a Kolmogorov scale for dissipation of G by equating the dissipation and the input of G,

$$\nu k_n^2G_n\sim\nu(-1)^nk_n^{7/2}\mathcal{R}[u_{n-1}u_nu_{-(n+1)}^*]\sim\overline{\eta}. \quad (24)$$

Using $u_n\sim(\overline{\varepsilon}/k_n)^{1/3}$ again gives

$$K_G\sim[\overline{\eta}/(\overline{\varepsilon}\nu)]^{2/5}. \quad (25)$$

The ratio of dissipation scales is then $K_G/K_E\sim\nu^{-2/5+3/4}=\nu^{7/20}$, so as for the case of helicity the small scales will have no net nonlinear flux of G. There is, however, a crucial difference between the dissipation of H and of G. The dissipation of helicity is forced to be of alternating signs whereas the dissipation of G can be of either sign at any shell. Thus one should expect the mean dissipation of G_n to vanish for $k_n>K_G$, and the dissipation would not determine the scaling of Π_n^G. So how would the scaling of Π_n^G be then? The nonlinear flux Π_n^G is constituted of terms of the form $k_n^{5/2}S_4(n)\sim k_n^{5/2}u_n^4\sim k_n^{7/6}$. This leading scaling is eliminated by detailed cancellations between terms in an inertial range. One would then expect K_G to represent a decorrelation scale where the

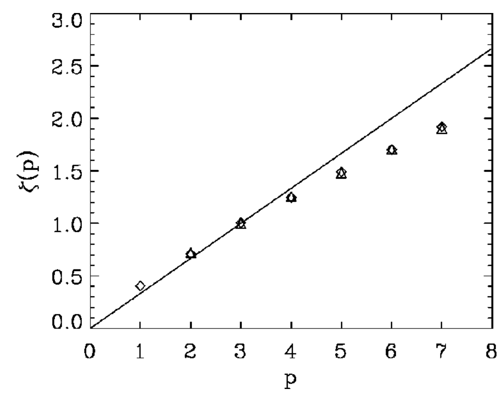

FIG. 2. The anomalous scaling exponents $\zeta(p)$ defined from $(\Pi_n^E/k_n)^{p/3}\sim k_n^{-\zeta(p)}$ for $(\epsilon,\lambda)=(1/g,g^2)$ (diamonds) coincide within the numerical uncertainty with the values found by L'vov *et al.* for $(\epsilon,\lambda)=(1/2,2)$ (triangles).

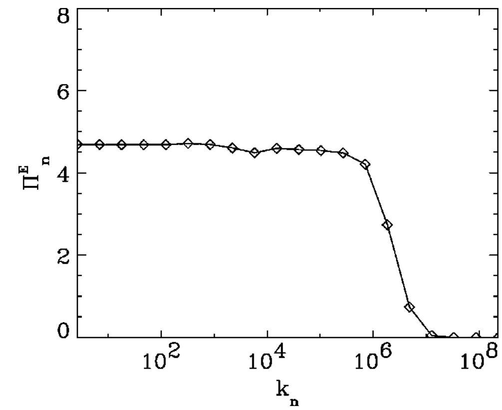

FIG. 3. The nonlinear flux of energy Π_n^E as a function of k_n.

individual terms become independent and the scaling of $|\Pi_n^G|$ becomes approximately that of the individual terms, which is $k_n^{7/6}$. This conjecture is tested in a numerical simulation.

XI. NUMERICAL SIMULATION

A simulation using a simple fourth-order Runge-Kutta scheme has been performed with the parameter values $(\epsilon,\lambda,\nu,N,f)=(1/g,g^2,10^{-9},20,(1+i)\delta_{1,n})$. The anomalous scaling exponents are defined from $\langle(\tilde{\Pi}_n^E)^{p/3}\rangle \sim k_n^{\zeta(p)}$. The results for this model are shown in Fig. 2 (diamonds) and coincide within the numerical accuracy of the simulation with the exponents found by L'vov *et al.* for the Sabra model with $(\epsilon,\lambda)=(1/2,2)$ (triangles). The nonlinear fluxes for E,H,G are shown in Figs. 3, 4, and 5. The fluxes for H and G fluctuate between positive and negative values, so that only the absolute values are shown. The scaling indicated by the straight lines confirms the conjectures made in the preceding section. However, even though the simulation is long enough and numerically accurate enough to determine the anomalous scaling exponents reliably, the nonlinear flux Π_n^G can only be determined using extreme numerical precision. If

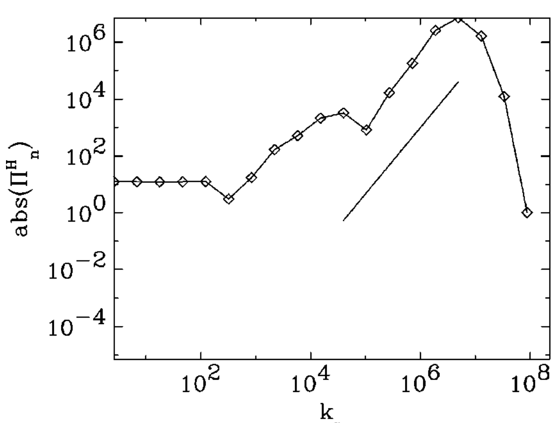

FIG. 4. The nonlinear flux of helicity $|\Pi_n^H|$ as a function of k_n. An inertial range with coexisting cascades of both energy and helicity is seen for the first few shells, after which the dissipation of helicity dominates the spectrum. The sign of the flux alternates for even and odd shells corresponding to dissipation of positive and negative helicity. The straight line indicates the scaling exponent 7/3 as is expected from the dissipation.

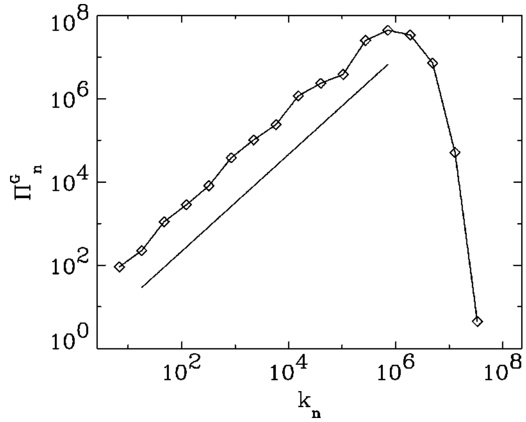

FIG. 5. The nonlinear flux of the third-order quantity Π_n^E as a function of k_n. The numerics has not converged and the scaling exponents 7/6 indicated by the line come from the standard deviation of the numerical value of the difference between correlators of the order $k_n^{7/6}$.

we conjecture that there is a constant flux of G of order unity $(\overline{\eta}\sim 1)$ through the inertial range, we must calculate Π_n^G as differences of fourth-order correlators of order $k_n^{7/6}$, which is about 10^6 at the end of the inertial range to obtain a constant of order unity. So in fact the graph in Fig. 5 does not reliably represent Π_n^G, it is merely numerical noise. To see this consider the average $|\langle\Pi_n^G\rangle(t)|=|\int_0^t \Pi_n^G(\tau)d\tau|$ as a function of t, see Fig. 6. The line is the curve $\sigma/\sqrt{t}$ which is expected for an independent random process. The standard deviation σ of the process is of the order $k_n^{7/6}$. The main justification for studying shell models is the possibility of accurate numerical calculations of correlators and scaling exponents for high Reynolds number flow. Here we see that even the shell model can been pushed to the limit where the determination of correlators by numerical simulation is impractical.

XII. SUMMARY

To summarize, it has been argued why this model is a natural choice for a shell model of turbulence. For the choice

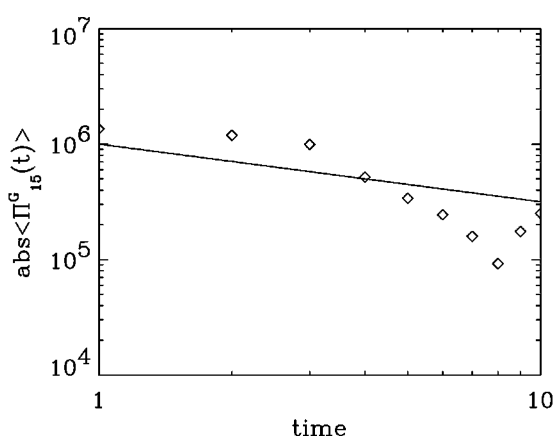

FIG. 6. The average $|\langle\Pi_n^G\rangle(t)|=|\int_0^t \Pi_n^G(\tau)d\tau|$ as a function of t. The straight line has the slope $-1/2$ as for an independent random variable. This shows that the quantity shown in Fig. 5 is dominated by noise. Note that the sampling time is long enough to determine the anomalous scaling exponents.

of parameters which conserves both energy, dimensionally correct helicity, and the third-order quantity G, the anomalous scaling exponents are the same as for the Sabra model, which conserves energy and dimensionally correct helicity but not G. So even though the model has a Hamiltonian structure, the nonpositive Hamiltonian, G, seems not relevant for determining the scaling properties of the model. It has been argued that there will not be an inertial range scaling regime for the fourth-order correlator associated with the flux of G. However, due to the extreme numerical accuracy required to determine this correlator, the numerical simulations presented here are not conclusive in the determination of this correlator.

ACKNOWLEDGMENTS

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